THE MATHEMATICS OF OPTIMIZATION

 Walter Nicholson, Microeconomic Theory - Basic Principles and Extensions, Thompson-Southwestern, 9th Edition, 2005, ISBN: 0324270860

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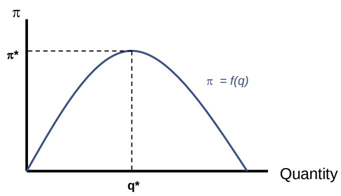
Maximization of a Function of One Variable

- Simple example: Manager of a firm wishes to maximize profits

$$\pi = f(q)$$



Maximum profits of π^* occur at q^*

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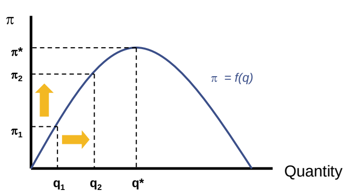
Maximization of a Function of One Variable

- The manager will likely try to vary q to see where the maximum profit occurs
  - an increase from q_1 to q_2 leads to a rise in π



$$\frac{\Delta\pi}{\Delta q} > 0$$

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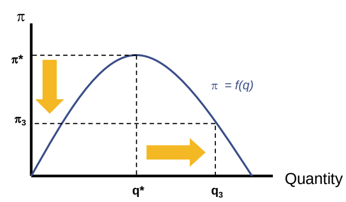
Maximization of a Function of One Variable

- If output is increased beyond q^*, profit will decline
  - an increase from q^* to q_3 leads to a drop in π



$$\frac{\Delta\pi}{\Delta q} < 0$$

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Basic Differentiation Rules

1. $\frac{d}{dx}(c) = 0 \quad (c \text{ is a constant})$

Ex. $f(x) = 5$

$f'(x) = 0$

2. $\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad (n \text{ is a real number})$

Ex. $f(x) = x^7$

$f'(x) = 7x^6$

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3. Sum-Difference Rule.

If f(x) = g(x) ± h(x), then f'(x) = g'(x) ± h'(x)

Define $u = g(x)$ $v = h(x)$ **then** $f(x) = u \pm v)$

Thus $f'(x) = \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

4. Product Rule:

If f(x) = g(x)h(x), then f'(x) = g(x)h'(x) + h(x)g'(x)

Define $u = g(x)$ $v = h(x)$ **then** $f(x) = uv)$

$$f'(x) = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

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5. Quotient Rule

If f(x) = $\frac{g(x)}{h(x)}$ ***then*** $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

$$f'(x) = \frac{[v\frac{du}{dx} - u\frac{dv}{dx}]}{v^2}$$

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Differentiation of Logarithmic Functions

Derivative of the Natural Logarithm

$$f'(x) = \frac{d}{dx}\ln x = \frac{1}{x} \quad (x \neq 0)$$

Generalized Rule for Natural Logarithm Functions

If *u* is a differentiable function, then

$$f'(x) = \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

$y = \ln 2x^2$

$f'(x) = \frac{4x}{2x^2} = \frac{2}{x}$

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The Chain Rule

If *f* is a differentiable function of *u* and *u* is a differentiable function of *x*, then the composite *f* (*u*) is a differentiable function of *x*, and

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} = \frac{df(u)}{du}\frac{du}{dx}$$

The derivative of a *f* (quantity) is the derivative of *f* evaluated at the quantity, times the derivative of the quantity.

$$TR = PQ(K,L)$$

$$\frac{dTR}{dL} = P\frac{dQ(K,L)}{dQ}\frac{dQ}{dL}$$

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Example of Profit Maximization

- Suppose that the relationship between profit and output is

$$\pi = 1{,}000q - 5q^2$$

- The first order condition for a maximum is

$$d\pi/dq = 1{,}000 - 10q = 0$$

$$q^* = 100$$

- Since the second derivative is always -10, $q = 100$ is a global maximum

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Functions of Several Variables

- Most goals of economic agents depend on several variables
  - trade-offs must be made
- The dependence of one variable (y) on a series of other variables ($x_1,x_2,\ldots,x_n$) is denoted by

$$y = f(x_1, x_2, \ldots, x_n)$$

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Partial Derivatives

- The partial derivative of y with respect to x_1 is denoted by

$$\frac{\partial y}{\partial x_1} \text{ or } \frac{\partial f}{\partial x_1} \text{ or } f_{x_1} \text{ or } f_1$$

- It is understood that in calculating the partial derivative, all of the other x's are held constant

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Calculating Partial Derivatives

1. If $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = 2ax_1 + bx_2 \quad \text{and}$$

$$\frac{\partial f}{\partial x_2} = f_2 = bx_1 + 2cx_2$$

2. If $y = f(x_1, x_2) = a\ln x_1 + b\ln x_2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = \frac{a}{x_1} \text{ and } \frac{\partial f}{\partial x_2} = f_2 = \frac{b}{x_2}$$

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Total Differential

- Suppose that $y = f(x_1, x_2, \ldots, x_n)$
- If all x's are varied by a small amount, the total effect on y will be

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n$$

$$dy = f_1dx_1 + f_2dx_2 + \ldots + f_ndx_n$$

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First-Order Condition for a Maximum (or Minimum)

- A necessary condition for a maximum (or minimum) of the function $f(x_1, x_2, \ldots, x_n)$ is that $dy = 0$ for any combination of small changes in the x's
- The only way for this to be true is if

$$f_1 = f_2 = \ldots = f_n = 0$$

- A point where this condition holds is called a critical point

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Constrained Maximization

- What if all values for the x's are not feasible?
  - the values of x may all have to be positive
  - a consumer's choices are limited by the amount of purchasing power available
- One method used to solve constrained maximization problems is the Lagrangian multiplier method

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Lagrangian Multiplier Method

- Suppose that we wish to find the values of $x_1, x_2, \ldots, x_n$ that maximize

$$y = f(x_1, x_2, \ldots, x_n)$$

subject to a constraint that permits only certain values of the x's to be used

$$g(x_1, x_2, \ldots, x_n) = 0$$

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Lagrangian Multiplier Method

- The Lagrangian multiplier method starts with setting up the expression

$$\mathbf{L} = f(x_1, x_2, \ldots, x_n) + \lambda g(x_1, x_2, \ldots, x_n)$$

where λ is an additional variable called a Lagrangian multiplier

- When the constraint holds, $\mathbf{L} = f$ because $g(x_1, x_2, \ldots, x_n) = 0$

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Lagrangian Multiplier Method

- First-Order Conditions

$$\partial\mathbf{L}/\partial x_1 = f_1 + \lambda g_1 = 0$$

$$\partial\mathbf{L}/\partial x_2 = f_2 + \lambda g_2 = 0$$

$$\vdots$$

$$\partial\mathbf{L}/\partial x_n = f_n + \lambda g_n = 0$$

$$\partial\mathbf{L}/\partial\lambda = g(x_1, x_2, \ldots, x_n) = 0$$

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Lagrangian Multiplier Method

- The first-order conditions can generally be solved for $x_1, x_2, \ldots, x_n$ and λ
- The solution will have two properties:
  - the x's will obey the constraint
  - these x's will make the value of **L** (and therefore f) as large as possible

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Lagrangian Multiplier Method

- The Lagrangian multiplier (λ) has an important economic interpretation
- The first-order conditions imply that

$$f_1/\text{-}g_1 = f_2/\text{-}g_2 = \ldots = f_n/\text{-}g_n = \lambda$$

  - the numerators above measure the marginal benefit that one more unit of x_i will have for the function f
  - the denominators reflect the added burden on the constraint of using more x_i

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Lagrangian Multiplier Method

- At the optimal choices for the x's, the ratio of the marginal benefit of increasing x_i to the marginal cost of increasing x_i should be the same for every x
- λ is the common cost-benefit ratio for all of the x's

$$\lambda = \frac{\text{marginal benefit of } x_i}{\text{marginal cost of } x_i}$$

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Lagrangian Multiplier Method

- If the constraint was relaxed slightly, it would not matter which x is changed
- The Lagrangian multiplier provides a measure of how the relaxation in the constraint will affect the value of y
- λ provides a "shadow price" to the constraint

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Lagrangian Multiplier Method

- A high value of λ indicates that y could be increased substantially by relaxing the constraint
  - each x has a high cost-benefit ratio
- A low value of λ indicates that there is not much to be gained by relaxing the constraint
- λ=0 implies that the constraint is not binding

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