Algebra and Calculus Review

Loyola University Maryland

1 Exponents

Given n a positive integer, x^n signifies that x is multiplied by itself n times. Here x is referred to as the **base** and n is termed an **exponent**. By convention, an exponent of 1 is not expressed: $x^1 = x$, $8^1 = 8$. By definition, any nonzero number or variable raised to the zero power is equal to 1: $x^0 = 1$, $3^0 = 1$. And 0^0 is undefined. Assuming a and b are positive integers and x and y are real numbers for which the following exist, the rules of exponents are outlined below:

- $x^a(x^b) = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $(x^a)^b = x^{ab}$
- $(xy)^a = x^a y^a$
- $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
- $\frac{1}{x^a} = x^{-a}$
- $\sqrt{x} = x^{\frac{1}{2}}$
- $\sqrt[a]{x} = x^{\frac{1}{a}}$
- $\sqrt[b]{x^a} = x^{\frac{a}{b}} = \left(x^{\frac{1}{b}}\right)^a$
- $x^{-\frac{a}{b}} = \frac{1}{x^{\frac{a}{b}}}$

2 Polynomials

Given an expression such as $5x^3$, x is called a **variable** because it can assume any number of different values and 5 is referred to as the **coefficient** of x. Expressions consisting simply of a real number or of a coefficient times one or more variables raised to the power of a positive integer are called **monomials**. Monomials can be added or subtracted to form **polynomials**. Each of the monomials comprising a polynomial is called a **term**. Terms that have the same variables and exponents are called **like terms**.

Like terms in polynomials can be added or subtracted by adding their coefficients. Unlike terms cannot be so added or subtracted.

- $4x^5 + 9x^5 = 13x^5$
- $(7x^3 + 5x^2 - 8x) + (11x^3 - 9x^2 - 2x) = 18x^3 - 4x^2 - 10x$
- $12xy - 3xy = 9xy$
- $(24x - 17y) + (6x + 5z) = 30x - 17y + 5z$

Like and unlike terms can be multiplied or divided by multiplying or dividing both the coefficients and variables.

- $(5x)(13y^2) = 65xy^2$
- $\frac{15x^4y^3z^6}{3x^2y^2z^3} = 5x^2yz^3$
- $(7x^3y^5)(4x^2y^4) = 28x^5y^9$
- $\frac{4x^2y^5z^3}{8x^5y^3z^4} = \frac{y^2}{2x^3z}$
- $(2x^3y)(17y^4z^2) = 34x^3y^5z^2$

3 Solving Linear Equations

A mathematical statement setting two algebraic expressions equal to each other is called an **equation**. An equation in which all variables are raised to the first power is known as a **linear equation**. A linear equation can be solved by moving the unknown variable to the left-hand side of the equal sign and all the other terms to the right-hand side.

The linear equation given below is solved in three easy steps:

$$\frac{x}{4} - 3 = \frac{x}{5} + 1$$

Step 1: Move all terms with the unknown variable x to the left hand side of the equal sign. In this example, subtract $\frac{x}{5}$ from both sides of the equation.

$$\frac{x}{4} - 3 - \frac{x}{5} = 1$$

Step 2: Move any terms without the unknown variable to the right hand side of the equal sign. In this example, add 3 to both sides of the equation.

$$\frac{x}{4} - \frac{x}{5} = 1 + 3 = 4$$

Step 3: Simplify both sides of the equation until the unknown variable is by itself on the left and the solution is on the right. In this example, multiply both sides of the equation by 20 and subtract.

$$20 \cdot \left(\frac{x}{4} - \frac{x}{5}\right) = 4 \cdot 20$$

$$
\begin{aligned}
5x - 4x &= 80 \\
x &= 80
\end{aligned}
$$

4 Simultaneous Equations

A system of simultaneous linear equations can be solved by either the substitution or elimination method.

Example 1 *The equilibrium conditions for two markets, butter and margarine, where P_b and P_m are the prices of butter and margarine, respectively, are given as follows:*

$$
\begin{aligned}
8P_b - 3P_m &= 7 \qquad (1) \\
-P_b + 7P_m &= 19 \qquad (2)
\end{aligned}
$$

These prices that will bring equilibrium to the model are found below by using the substitution and elimination methods.

4.1 Substitution Method

Step 1: Solve one of the equations for one variable in terms of the other. Solving Equation (2) for P_b gives

$$P_b = 7P_m - 19$$

Step 2: Substitute the value of that term into the other equation $\big($in this case, Equation (1)$\big)$ and solve for P_m

$$
\begin{aligned}
8P_b - 3P_m &= 7 \\
8(7P_m - 19) - 3P_m &= 7 \\
56P_m - 152 - 3P_m &= 7 \\
53P_m &= 159 \\
P_m &= 3
\end{aligned}
$$

Step 3: Substitute $P_m = 3$ in either Equation (1) or Equation (2) to find P_b.

$$
\begin{aligned}
8P_b - 3(3) &= 7 \\
8P_b &= 16 \\
P_b &= 2
\end{aligned}
$$

4.2 Elimination Method

Step 1: Multiply Equation (1) by the coefficient of P_b (or P_m) in Equation (2) and Equation (2) by the coefficient of P_b (or P_m) in Equation (1). Picking P_m, we get

$$\begin{aligned} 7(8P_b - 3P_m &= 7) \Rightarrow 56P_b - 21P_m = 49 \quad (3) \\ -3(-P_b + 7P_m &= 19) \Rightarrow 3P_b - 21P_m = -57 \quad (4) \end{aligned}$$

Step 2: Subtract 4 from 3 to eliminate the selected variable.

$$\begin{aligned} 53P_b &= 106 \\ P_b &= 2 \end{aligned}$$

Step 3: Substitute $P_b = 2$ in Equation (3) or Equation (4) to find P_m as in step 3 of the substitution method.

5 Functions

A **function** f is a rule which assigns to each value of a variable (x), called the **argument** of the function, one and only one value $[f(x)]$, referred to as the value of the function at x. The **domain** of a function refers to the set of all possible values of x; the **range** is the set of all possible values of $f(x)$. Sometimes letters other than f are used to express functions. These include g, h, or Greek letters (π, ϕ, etc.). Functions encountered frequently in economics include:

Linear function: $f(x) = mx + b$
Quadratic function: $f(x) = ax^2 + bx + c \qquad (a \neq 0)$
Polynomial function of degree n: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$
Note: n = nonnegative integer; $a_n \neq 0$
Rational function: $f(x) = \frac{g(x)}{h(x)}$
Note: $g(x)$ and $h(x)$ are both polynomials and $h(x) \neq 0$. Rational comes from "ratio."
Power function: $f(x) = ax^n \qquad$ (n = any real number)

6 Graphs, Slopes, and Intercepts

In graphing a function such as $y = f(x)$, x is placed on the horizontal axis and is known as the **independent variable**, y is placed on the vertical axis and is called the **dependent variable**. The graph of a linear function is a straight line. The **slope** of a line measures the change in y (Δy) divided by a change in x (Δx). The slope indicates the steepness and direction of a line. the greater the absolute value of the slope, the steeper the line. A positively sloped line moves up from left to right; a negatively sloped line moves down. The slope of a horizontal line, for which $\Delta y = 0$, is zero. The slope of a vertical line, for which $\Delta x = 0$, is undefined (it does not exist because division by zero is impossible). The **y intercept** is the point where the graph crosses the y-axis; it occurs when $x = 0$. The **x intercept** is the point where the line intersects the x-axis; it occurs when $y = 0$.

For a line passing through points (x_1, y_1) and (x_2, y_2), the slope m is calculated as follows:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2$$

7 The Derivative and the Rules of Differentiation

Given a function $y = f(x)$, the **derivative** of the function, $f'(x)$, is itself a function which measures both the slope and the instantaneous rate of change of the original function $f(x)$ at a given point. $f'(x)$ is read "the derivative of f with respect to x" or "f prime of x." $f'(x)$ can also be referred to as the first derivative or the first-order derivative. The derivative of a function can be written in many different ways. If $y = f(x)$, the derivatives can be expressed as any of the following:

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}[f(x)] \qquad D_x[f(x)]$$

If the derivative of $y = f(x)$ is evaluated at $x = a$, proper notation includes $f'(a)$ and $\left.\frac{dy}{dx}\right|_a$. A function is **differentiable** at a point if the derivative exists at that point. To be differentiable at a point, a function must (1) be continuous at that point and (2) have a unique tangent at that point. Note that continuity alone does not ensure differentiability. It can be the case that $f(x)$ is continuous at a point b, but is not differentiable at b because there is a sharp point (or kink) such that an infinite number of tangent lines (and no one unique tangent line) can be drawn.

7.1 Rules of Differentiation

Differentiation is the process of finding the derivative of a function. It involves nothing more complicated than applying a few basic rules or formulas to a given function. In explaining the rules of differentiation for a function such as $y = f(x)$, other functions such as $g(x)$ and $h(x)$ are commonly used, where g and h are both unspecified functions of x. The rules of differentiation are listed below:

Constant Function Rule: The derivative of a constant function $f(x) = k$, where k is a constant, is zero.

Given $f(x) = k, f'(x) = 0$
Example: Given $f(x) = 8, f'(x) = 0$
Example: Given $f(x) = -6, f'(x) = 0$

Linear Function Rule: The derivative of a linear function $f(x) = mx + b$ is equal to m, the coefficient of x. The derivative of a variable raised to the first power is always equal to the coefficient of the variable, while the derivative of a constant is simply zero.

Given $f(x) = mx + b, f'(x) = m$
Example: Given $f(x) = 3x + 2, f'(x) = 3$
Example: Given $f(x) = 5 - \frac{1}{4}x, f'(x) = -\frac{1}{4}$
Example: Given $f(x) = 12x, f'(x) = 12$

Power Function Rule: The derivative of a power function $f(x) = kx^n$, where k is a constant and n is any real number, is equal to the coefficient k times the exponent n, multiplied by the variable x raised to the $n - 1$ power.

Given $f(x) = kx^n, f'(x) = k \cdot n \cdot x^{n-1}$
Example: Given $f(x) = 4x^3, f'(x) = 4 \cdot 3 \cdot x^{3-1} = 12x^2$
Example: Given $f(x) = 5x^2, f'(x) = 5 \cdot 2 \cdot x^{2-1} = 10x$
Example: Given $f(x) = x^4, f'(x) = 1 \cdot 4 \cdot x^{4-1} = 4x^3$

Rules for Sums and Differences: The derivative of a sum of two functions $f(x) = g(x) + h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions, is equal to the sum of the derivatives of the individual functions. Similarly, the derivative of the difference of two functions is equal to the difference of the derivatives of the two functions.

Given $f(x) = g(x) \pm h(x), f'(x) = g'(x) \pm h'(x)$

Example: Given $f(x) = 12x^5 - 4x^4, f'(x) = 60x^4 - 16x^3$

Example: Given $f(x) = 9x^2 + 2x - 3, f'(x) = 18x + 2$

Product Rule: The derivative of a product $f(x) = g(x) \cdot h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions, is equal to the first function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first.

Given $f(x) = g(x) \cdot h(x), f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$

Example: Given $f(x) = 3x^4(2x - 5)$, let $g(x) = 3x^4$ and $h(x) = 2x - 5$

Step 1: Take the individual derivatives to get $g'(x) = 12x^3$ and $h'(x) = 2$

Step 2: Substitute these values in the product-rule formula to get

$$f'(x) = 3x^4(2) + (2x - 5)(12x^3)$$

Step 3: Simplify algebraically to get $f'(x) = 6x^4 + 24x^4 - 60x^3 = 30x^4 - 60x^3$

Quotient Rule: The derivative of a quotient $f(x) = g(x) \div h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions and $h(x) \neq 0$, is equal to denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the denominator squared.

Given $f(x) = \frac{g(x)}{h(x)}, f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$

Example: Given $f(x) = \frac{5x^3}{4x+3}$, let $g(x) = 5x^3$ and $h(x) = 4x + 3$

Step 1: Take the individual derivatives to get $g'(x) = 15x^2$ and $h'(x) = 4$

Step 2: Substitute these values in the quotient-rule formula to get

$$f'(x) = \frac{(4x+3)(15x^2) - 5x^3(4)}{(4x+3)^2}$$

Step 3: Simplify algebraically to get $f'(x) = \frac{60x^3 + 45x^2 - 20x^3}{(4x+3)^2} = \frac{40x^3 + 45x^2}{(4x+3)^2} = \frac{5x^2(8x+9)}{(4x+3)^2}$

Generalized Power Function Rule: The derivative of a function raised to a power, $f(x) = [g(x)]^n$, where $g(x)$ is a differentiable function and n is any real number, is equal to the exponent n times the function $g(x)$ raised to the $n - 1$ power, multiplied in turn by the derivative of the function itself $g'(x)$.

Given $f(x) = [g(x)]^n, f'(x) = n[g(x)]^{n-1} \cdot g'(x)$

Example: Given $f(x) = (x^3+6)^5$, let $g(x) = x^3+6$

Step 1: Take the individual derivative to get $g'(x) = 3x^2$

Step 2: Substitute these values into the generalized power function formula to get

$f'(x) = 5(x^3+6)^{5-1} \cdot 3x^2$

Step 3: Simplify algebraically to get $f'(x) = 5(x^3+6)^4 \cdot 3x^2 = 15x^2(x^3+6)^4$

Chain Rule: Given a **composite function**, also called a **function of a function**, in which y is a function of u and u in turn is a function of x, that is, $y = f(u)$ and $u = g(x)$, then $y = f[g(x)]$ and the derivative of y with respect to x is equal to the derivative of the first function with respect to u times the derivative of the second function with respect to x.

Given $y = f[g(x)], \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example: Given $y = (5x^2+3)^4$, let $y = u^4$ and $u = 5x^2+3$

Step 1: Take the individual derivatives to get $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 10x$

Step 2: Substitute these values into the chain rule formula to get

$\frac{dy}{dx} = 4u^3 \cdot 10x = 40xu^3$

Step 3: Express the derivative in terms of a single variable by substituting

$u = 5x^2+3$ to get $\frac{dy}{dx} = 40x(5x^2+3)^3$

For more complicated functions, different combinations of the basic rules must be used.

7.2 Second-Order Derivatives

The second-order derivative, written $f''(x)$, measures the slope and the rate of change of the first derivative, just as the first derivative measures the slope and the rate of change of the original or **primitive function**. $f''(x)$ can also be referred to as the second derivative. Given $y = f(x)$, common notation for the second-order derivative includes $f''(x)$, $\frac{d^2y}{dx^2}$, y'', and D^2y. Second-order derivatives are found by applying the rules of differentiation to the first-order derivative.

Example: Given $f(x) = 2x^4+5x^3+3x^2$

Step 1: Find the first-order derivative to get $f'(x) = 8x^3+15x^2+6x$

Step 2: Differentiate $f'(x)$ with respect to x to get $f''(x) = 24x^2+30x+6$

8 Uses of the Derivative in Mathematics and Economics

8.1 Increasing and Decreasing Functions

A function $f(x)$ is said to be **increasing** (**decreasing**) at $x = a$ if in the immediate vicinity of the point $[a, f(a)]$ the graph of the function rises (falls) as it moves from left to right. Since the first derivative measures the rate of change and slope of a function, a positive first derivative at $x = a$ indicates the function is increasing at a; a negative first derivative indicates it is decreasing.

In short,
Increasing function at $x = a$ if $f'(a) > 0$
Decreasing function at $x = a$ if $f'(a) < 0$

A function that increases (or decreases) over its entire domain is called a **monotonic function**. It is said to increase (decrease) **monotonically**.

8.2 Concavity and Convexity

A function $f(x)$ is **concave** (or "**concave down**") at $x = a$ if in some small region close to the point $[a, f(a)]$ the graph of the function lies completely below its tangent line. A function is **convex** (or "**concave up**") at $x = a$ if in an area very close to $[a, f(a)]$ the graph of the function lies completely above its tangent line. A positive second derivative at $x = a$ denotes the function is convex at $x = a$; a negative second derivative at $x = a$ denotes the function is concave at $x = a$. The sign of the first derivative is irrelevant for concavity.

In short,
$f(x)$ is convex at $x = a$ if $f''(a) > 0$
$f(x)$ is concave at $x = a$ if $f''(a) < 0$

If $f''(x) > 0$ for all x in the domain, $f(x)$ is **strictly convex**. If $f''(x) < 0$ for all x in the domain, $f(x)$ is **strictly concave**.

8.3 Optimization of Functions

Optimization is the process of finding the relative maximum or minimum of a function. There are two steps to optimize a function.

Step 1: Take the first derivative, set it equal to zero, and solve for the critical point(s). This step represents a necessary condition known as the **first-order condition**. It identifies all the points at which the function is neither increasing nor decreasing, but at a plateau. All such points are candidates for a possible relative maximum or minimum.

Step 2: Take the second derivative, evaluate it at the critical point(s), and check the sign(s). If at a critical point a,

$f''(a) < 0$, then the function is concave at a, and hence at a relative maximum.

$f''(a) > 0$, then the function is convex at a, and hence at a relative minimum.

$f''(a) = 0$, then the test is inconclusive.

Assuming the necessary first-order condition is met, this step, known as the **second-order derivative test**, or simply the **second-order condition**, represents a sufficiency condition.

In short,
Relative maximum: $f'(a) = 0$ and $f''(a) < 0$
Relative minimum: $f'(a) = 0$ and $f''(a) > 0$

Note that if the function is strictly concave (convex), then there will be only one maximum (minimum) called a global maximum (minimum).

Example 2 *Optimize* $f(x) = 2x^3 - 30x^2 + 126x + 59$.

Step 1: Find the critical points by taking the first derivative, setting it equal to zero, and solving for x.

$$\begin{aligned} f'(x) = 6x^2 - 60x + 126 &= 0 \\ 6(x-3)(x-7) &= 0 \end{aligned}$$

There are two critical points: $x = 3$ and $x = 7$.

Step 2: Test for concavity by taking the second derivative, evaluating it at the critical points, and checking the signs to distinguish between a relative maximum and minimum.

$$\begin{aligned} f''(x) &= 12x - 60 \\ f''(3) &= 12(3) - 60 = -24 \\ f''(7) &= 12(7) - 60 = 24 \end{aligned}$$

Since $f''(3) = -24 < 0$, then the function is maximized at $x = 3$. Since $f''(7) = 24 > 0$, then the function is minimized at $x = 7$.

8.4 Marginal Concepts

Marginal cost in economics is defined as the change in total cost incurred from the production of an additional unit. **Marginal revenue** is defined as the change in total revenue brought about by the sale of an extra good. Since total cost (TC) and total revenue (TR) are both functions of the level of output (Q), marginal cost (MC) and marginal revenue (MR) can each be expressed mathematically as the derivatives of their respective total functions.

$$\begin{aligned} MC &= \frac{dTC}{dQ} \\ MR &= \frac{dTR}{dQ} \end{aligned}$$

Example: If $TR = 75Q - 4Q^2$, then $MR = \frac{dTR}{dQ} = 75 - 8Q$
Example: If $TC = Q^2 + 7Q + 23$, then $MC = \frac{dTC}{dQ} = 2Q + 7$
Example: Given the inverse demand function $P = 30 - 2Q$, the marginal revenue function can be found by first finding the total revenue function and then taking the derivative of that function with respect to Q.

Step 1: $TR = PQ = (30 - 2Q)Q = 30Q - 2Q^2$
Step 2: $MR = \frac{dTR}{dQ} = 30 - 4Q$

8.5 Optimizing Economic Functions

The economist is frequently called upon to help a firm maximize profits and levels of physical output and productivity, as well as to minimize costs, levels of pollution, and the use of scarce natural resources. This is done with the help of techniques developed in this document.

Example 3 *Maximize profits (π) for a firm, given total revenue (TR) =* $4000Q - 33Q^2$ *and total cost (TC) =* $2Q^3 - 3Q^2 + 400Q + 5000$*, assuming* $Q > 0$.

Step 1: Set up the profit function ($\pi = TR - TC$)

$$\begin{aligned} \pi &= 4000Q - 33Q^2 - (2Q^3 - 3Q^2 + 400Q + 5000) \\ &= -2Q^3 - 30Q^2 + 3600Q - 5000 \end{aligned}$$

Step 2: Take the first derivative, set it equal to zero, and solve for Q to find the critical points.

$$\begin{aligned} \pi' &= -6Q^2 - 60Q + 3600 = 0 \\ &= -6(Q^2 + 10Q - 600) = 0 \\ &= -6(Q + 30)(Q - 20) = 0 \end{aligned}$$

There are two critical points: $Q = -30$ and $Q = 20$. However, we can disregard $Q = -30$ since it does not make sense for quantity to take on a negative value. This is why we previously assumed that $Q > 0$. Note that a quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$, can be solved using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Step 3: Take the second derivative, evaluate it at the positive critical point, and check the sign for concavity to be sure of a relative maximum (Remember: we assume that we want to maximize profits).

$$\begin{aligned} \pi'' &= -12Q - 60 \\ &= 12(20) - 60 = -300 \end{aligned}$$

Since $\pi''(20) = -300 < 0$, then profit is maximized at $Q = 20$. In fact, profit (π) = $-2(20)^3 - 30(20)^2 + 3600(20) - 5000 = \$39,000$.

9 Calculus of Multivariable Function

9.1 Functions of Several Variables and Partial Derivatives

Study of the derivative in the previous section was limited to functions of a single independent variable such as $y = f(x)$. Many economic activities, however, involve functions of more than one independent variable. $z = f(x, y)$ is defined as a **function of two independent variables** if there exists one and only one value of z in the range of f for each ordered pair of real numbers (x, y) in the domain of f. By convention, z is the **dependent variable**; x and y are the **independent variables**.

To measure the effect of a change in a single independent variable (x or y) on the dependent variable (z) in a multivariable function, the **partial derivative** is needed. The partial derivative of z with respect to x measures the instantaneous rate of change of z with respect to x while y is held constant. It can be written in a number of ways, including $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x$, or z_x. The partial derivative of z with respect to y measures the rate of change of z with respect to y while x is held constant. It can be written in a number of ways, including $\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), f_y$, or z_y.

Partial differentiation with respect to one of the independent variables follows the same rules as ordinary differentiation while the other independent variables are treated as constant.

Example 4 *Find the partial derivatives of $z = 3x^2y^3$ with respect to both x and y.*

Step 1: When differentiating with respect to x, treat the y term as a constant by mentally bracketing it with the coefficient:

$$z = [3y^3] \cdot x^2$$

Step 2: Take the derivative of the x term, holding the y term constant,

$$\begin{aligned} \frac{\partial z}{\partial x} = z_x &= [3y^3] \cdot \frac{d}{dx}(x^2) \\ &= [3y^3] \cdot 2x \end{aligned}$$

Step 3: Multiply and rearrange terms to obtain

$$\frac{\partial z}{\partial x} = z_x = 6xy^3$$

Step 4: When differentiating with respect to y, treat the x term as a constant by bracketing it with the coefficient; then take the derivative as was done above:

$$
\begin{aligned}
z &= [3x^2] \cdot y^3 \\
\frac{\partial z}{\partial y} = z_y &= [3x^2] \cdot \frac{d}{dy}(y^3) \\
&= [3x^2] \cdot 3y^2 \\
&= 9x^2y^2
\end{aligned}
$$

9.2 Rules of Partial Differentiation

Partial derivatives follow the same basic patterns as the rules of differentiation in Section 7.1. A few key rules are given below.

Product Rule: Given $z = g(x,y) \cdot h(x,y)$,

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= g(x,y) \cdot \frac{\partial h}{\partial x} + h(x,y) \cdot \frac{\partial g}{\partial x} \\
\frac{\partial z}{\partial y} &= g(x,y) \cdot \frac{\partial h}{\partial y} + h(x,y) \cdot \frac{\partial g}{\partial y}
\end{aligned}
$$

Example: Given $z = (3x+5)(2x+6y)$,

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= (3x+5)(2) + (2x+6y)(3) = 12x + 10 + 18y \\
\frac{\partial z}{\partial y} &= (3x+5)(6) + (2x+6y)(0) = 18x + 30
\end{aligned}
$$

Quotient Rule: Given $z = \frac{g(x,y)}{h(x,y)}$ and $h(x,y) \neq 0$,

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2} \\
\frac{\partial z}{\partial y} &= \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}
\end{aligned}
$$

Example: Given $z = \frac{6x+7y}{5x+3y}$,

$$
\frac{\partial z}{\partial x} = \frac{(5x+3y)(6) - (6x+7y)(5)}{(5x+3y)^2}
$$

$$
\begin{aligned}
&= \frac{30x+18y-30x-35y}{(5x+3y)^2} \\
&= \frac{-17y}{(5x+3y)^2} \\
\frac{\partial z}{\partial y} &= \frac{(5x+3y)(7)-(6x+7y)(3)}{(5x+3y)^2} \\
&= \frac{35x+21y-18x-21y}{(5x+3y)^2} \\
&= \frac{17x}{(5x+3y)^2}
\end{aligned}
$$

Generalized Power Function Rule: Given $z = [g(x,y)]^n$,

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= n[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial x} \\
\frac{\partial z}{\partial y} &= n[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial y}
\end{aligned}
$$

Example: Given $z = (x^3+7y^2)^4$,

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= 4(x^3+7y^2)^3 \cdot (3x^2) = 12x^2(x^3+7y^2)^3 \\
\frac{\partial z}{\partial y} &= 4(x^3+7y^2)^3 \cdot (14y) = 56y(x^3+7y^2)^3
\end{aligned}
$$

9.3 Second-Order Partial Derivatives

Given a function $z = f(x,y)$, the **second-order partial derivative** signifies that the function has been differentiated partially with respect to one of the independent variables twice while the other independent variable has been held constant:

$$
f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}
$$

In effect, f_{xx} measures the rate of change of the first-order partial derivative f_x with respect to x while y is held constant. And f_{yy} is exactly parallel.

The **cross partial derivatives** f_{xy} and f_{yx} indicate that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative has in turn been partially differentiated with respect to the other independent variable:

$$
f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 y}{\partial x \partial y}
$$

In effect, a cross partial measures the rate of change of a first-order partial derivative with respect to the other independent variable. Notice how the order of the independent variables changes in the different forms of notation. If both cross partial derivatives are continuous, then they will be identical.

Example: Take the first, second, and cross partial derivatives for $z = 7x^3 + 9xy + 2y^5$.

First partial derivatives: $\frac{\partial z}{\partial x} = z_x = 21x^2 + 9y$

$\frac{\partial z}{\partial y} = z_y = 9x + 10y^4$

Second partial derivatives: $\frac{\partial^2 z}{\partial x^2} = z_{xx} = 42x$

$\frac{\partial^2 z}{\partial y^2} = z_{yy} = 40y^3$

Cross partial derivatives: $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(21x^2 + 9y) = z_{xy} = 9$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(9x + 10y^4) = z_{yx} = 9$

9.4 Constrained Optimization with Lagrange Multiplier

Differential calculus is also used to maximize or minimize a function subject to a constraint. Given a function $f(x,y)$ subject to a constraint $g(x,y) = k$, where k is a constant, a new function F can be formed by (1) setting the constraint equal to zero, (2) multiplying it by λ (the **Lagrange multiplier**), and (3) adding the product to the original function:

$$\mathscr{L}(x,y,\lambda) = f(x,y) + \lambda[k - g(x,y)]$$

$\mathscr{L}(x,y,\lambda)$ is the **Lagrangian function**, $f(x,y)$ is the original or **objective function**, and $g(x,y)$ is the **constraint**. Since the constraint is always set equal to zero, the product $\lambda[k - g(x,y)]$ also equals zero, and the addition of the term does not change the value of the objective function. Critical values x_0, y_0, and λ_0, at which the function is optimized, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero, and solving simultaneously:

$$\mathscr{L}_x(x,y,\lambda) = 0 \qquad \mathscr{L}_y(x,y,\lambda) = 0 \qquad \mathscr{L}_\lambda(x,y,\lambda) = 0$$

Example 5 *Optimize the function* $f(x,y) = 4x^2 + 3xy + 6y^2$ *subject to the constraint* $x + y = 56$.

Step 1: Set the constraint equal to zero by subtracting the variables from the constant

$$56 - x - y = 0$$

Step 2: Multiply this difference by λ and add the product of the two to the objective function in order to form the Lagrangian function $\mathscr{L}$

$$\mathscr{L} = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Step 3: Take the first-order partials, set them equal to zero, and solve simultaneously

$$\begin{aligned} \mathscr{L}_x &= 8x + 3y - \lambda = 0 \\ \mathscr{L}_y &= 3x + 12y - \lambda = 0 \\ \mathscr{L}_\lambda &= 56 - x - y = 0 \end{aligned}$$

Step 4: Subtract $\mathscr{L}_y$ from $\mathscr{L}_x$ to eliminate λ gives

$$5x - 9y = 0 \Rightarrow x = 1.8y$$

Step 5: Substitute $x = 1.8y$ into $\mathscr{L}_\lambda$

$$56 - 1.8y - y = 0 \Rightarrow y_0 = 20$$

Step 6: Plug in y_0 into $x = 1.8y$ to get

$$x = 1.8(20) \Rightarrow x_0 = 36$$

Step 7: Plug in x_0 and y_0 into either $8x + 3y - \lambda = 0$ or $3x + 12y - \lambda = 0$

If using $8x + 3y - \lambda = 0$

$$\begin{aligned} \lambda &= 8x + 3y \\ &= 8(36) + 3(20) \\ \lambda_0 &= 348 \end{aligned}$$

If using $3x + 12y - \lambda = 0$

$$\begin{aligned} \lambda &= 3x12y \\ &= 3(36) + 12(20) \\ \lambda_0 &= 348 \end{aligned}$$

Notice that you get the same value of λ_0 regardless of which first-order condition you use.

Step 8: Substitute the critical values (x_0, y_0, and λ_0) into $\mathscr{L}$

$$\begin{aligned}
\mathscr{L} &= 4(36)^2+3(36)(20)+6(20)^2+(348)(56-36-20) \\
&= 4(1296)+3(720)+6(400)+348(0) \\
&= 9744
\end{aligned}$$

Notice that at the critical values, the Lagrangian function $\mathscr{L}$ equals the objective function $f(x,y)$ because the constraint equals zero.

9.5 Significance of the Lagrange Multiplier

The Lagrange multiplier λ approximates the marginal impact on the objective function caused by a small change in the constant of the constraint. With $\lambda = 348$ in Example 5, for instance, a one-unit increase (decrease) in the constant of the constraint would cause $\mathscr{L}$ to increase (decrease) by approximately 348 units. Lagrange multipliers are often referred to as **shadow prices**. In utility maximization subject to a budget constraint, λ will estimate the marginal utility of an extra dollar of income.

Note that $\lambda[k-g(x,y)] = \lambda[g(x,y)-k] = 0$ in the Lagrangian function $\mathscr{L}(x,y,\lambda)$. As such, either form can be added to or subtracted from the objective function without changing the critical values of x and y. Only the sign of λ will be affected. However, for the interpretation of λ given in this subsection to be valid, the precise form of $\mathscr{L}(x,y,\lambda) = f(x,y)+\lambda[k-g(x,y)]$ should be adhered to.

10 Exponential and Logarithmic Functions

10.1 Exponential Functions

An **exponential function** is a function in which a constant base a is raised to a variable exponent x. Mathematically, it is defined as

$$y = a^x \qquad a > 0 \qquad a \neq 1$$

Commonly used to express rates of growth and decay, such as compound interest and depreciation, exponential functions have the following general properties. Given $y = a^x$, $a > 0$, and $a \neq 1$:

- The domain of the function is the set of all real numbers; the range of the function is the set of all positive real numbers.
- For $a > 1$, the function is increasing and convex; for $0 < a < 1$, the function is decreasing and convex.
- At $x = 0$, $y = 1$ regardless of the base.

10.2 Logarithmic Functions

Interchanging the variables of an exponential function f defined by $y = a^x$ gives rise to a new function g defined by $x = a^y$ such that any ordered pair of numbers in f will also be found in g in reverse order. For example, if $f(2) = 4$, then $g(4) = 2$; if $f(3) = 8$, then $g(8) = 3$. The new function g, the inverse of the exponential function f, is called a **logarithmic function** with base a. Instead of $x = a^y$, the logarithmic function with base a is more commonly written as

$$y = \log_a x \qquad a > 0, a \neq 1$$

$\log_a x$ is the exponent to which a must be raised to get x. Any positive number except 1 may serve as the base for a logarithm. The common logarithm of x, written $\log_{10} x$ or simply $\log x$, is the exponent to which 10 must be raised to get x. Logarithms have the following properties. Given $y = \log_a x$, $a > 0$, and $a \neq 1$:

- The domain of the function is the set of all positive real numbers; the range is the set of all real numbers. Note that this is the exact opposite of the exponential function.
- For base $a > 1$, $f(x)$ is increasing and concave. For $0 < a < 1$, $f(x)$ is decreasing and convex.
- At $x = 1$, $y = 0$ regardless of the base.

10.3 Properties of Exponents and Logarithms

Assuming $a, b > 0$; $a, b \neq 1$; and x and y are any real numbers:

- $a^x \cdot a^y = a^{x+y}$
- $(a^x)^y = a^{xy}$
- $\frac{1}{a^x} = a^{-x}$
- $a^x \cdot b^x = (ab)^x$
- $\frac{a^x}{a^y} = a^{x-y}$
- $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

For a, x, and y positive numbers, n a real number, and $a \neq 1$:

- $\log_a xy = \log_a x + \log_a y$
- $\log_a x^n = n \log_a x$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$

10.4 Natural Exponential and Logarithmic Functions

The most commonly used base for exponential and logarithmic functions is the irrational number e. Exponential functions to base e are called **natural exponential functions** and are written $y = e^x$; logarithmic functions to base e are termed **natural logarithmic functions** and are expressed as $y = \log_e x$ or, more commonly, $\ln x$. Thus, $\ln x$ is simply the exponent or power to which e must be raised to get x.

10.5 Solving Natural Exponential and Logarithmic Functions

Since natural exponential functions and natural logarithmic functions are inverses of each other, one is generally helpful in solving the other. Mindful that $\ln x$ signifies the power to which e must be raised to get x, it follows that:

- e raised to the natural log of a constant ($a > 0$), a variable ($x > 0$), or a function of a variable $[f(x) > 0]$ must be equal to that constant, variable, or function of the variable:

$$e^{lna} = a \qquad e^{lnx} = x \qquad e^{lnf(x)} = f(x)$$

- Conversely, the natural log of e raised to the power of a constant, variable, or function of a variable must also equal that constant, variable, or function of the variable:

$$\ln e^a = a \qquad \ln e^x = x \qquad \ln e^{f(x)} = f(x)$$

Example: Solve for x given $5e^{x+2} = 120$

Step 1: Solve algebraically for e^{x+2},

$$
\begin{aligned}
5e^{x+2} &= 120 \\
e^{x+2} &= 24
\end{aligned}
$$

Step 2: Take the natural log of both sides to eliminate e

$$
\begin{aligned}
\ln e^{x+2} &= \ln 24 \\
x+2 &= \ln 24 \\
x &= \ln 24 - 2
\end{aligned}
$$

Example: Solve for x given $6\ln x - 7 = 12.2$

Step 1: Solve algebraically for $\ln x$

$$
\begin{aligned}
6\ln x &= 19.2 \\
\ln x &= 3.2
\end{aligned}
$$

Step 2: Set both sides of the equation as exponents of e to eliminate the natural log expression,

$$
\begin{aligned}
e^{\ln x} &= e^{3.2} \\
x &= e^{3.2}
\end{aligned}
$$

10.6 Logarithmic Transformation of Nonlinear Functions

Some nonlinear functions such as Cobb-Douglas production functions, can easily be converted to linear functions through simple logarithmic transformation. For example, we can use the properties of logarithms (see Section 10.3) to transform a generalized Cobb-Douglas production function into a function that is log-linear:

$$
\begin{aligned}
q &= AK^{\alpha}L^{\beta} \\
\ln q &= \ln A + \alpha \ln K + \beta \ln L
\end{aligned}
$$

11 Differentiation of Exponential and Logarithmic Functions

11.1 Rules of Differentiation

The rules of exponential and logarithmic differentiation are presented below.

Natural Exponential Function Rule: Given $f(x) = e^{g(x)}$, where $g(x)$ is a differentiable function of x, the derivative is

$$f'(x) = e^{g(x)} \cdot g'(x) \tag{5}$$

In short, the derivative of a natural exponential function is equal to the original natural exponential function times the derivative of the exponent.

Example: Given $f(x) = e^x$, let $g(x) = x$

Step 1: Take the individual derivative to find $g'(x) = 1$

Step 2: Substituting this value into Equation (5) to get

$$f'(x) = e^x \cdot 1 = e^x$$

Note: The derivative of e^x is simply e^x, the original function itself.

Example: Given $f(x) = e^{x^2}$, let $g(x) = x^2$

Step 1: Take the individual derivative to find $g'(x) = 2x$

Step 2: Substituting this value into Equation (5) to get

$$f'(x) = e^{x^2} \cdot 2x = 2xe^{x^2}$$

Example: Given $f(x) = 3e^{7-2x}$, let $g(x) = 7 - 2x$

Step 1: Take the individual derivative to find $g'(x) = -2$

Step 2: Substituting this value into Equation (5) to get

$$f'(x) = 3e^{7-2x} \cdot -2 = -6e^{7-2x}$$

Natural Logarithmic Function Rule: Given $f(x) = \ln g(x)$, where $g(x)$ is positive and differentiable, the derivative is

$$f'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)} \tag{6}$$

Example: Given $f(x) = \ln x$, let $g(x) = x$

Step 1: Take the individual derivative to find $g'(x) = 1$

Step 2: Substituting this value into Equation (6) to get

$$f'(x) = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

Example: Given $f(x) = \ln 6x^2$, let $g(x) = 6x^2$

Step 1: Take the individual derivative to find $g'(x) = 12x$

Step 2: Substituting this value into Equation (6) to get

$$f'(x) = \frac{1}{6x^2} \cdot 12x = \frac{2}{x}$$

Example: Given $f(x) = \ln(x^2 + 6x + 2)$, let $g(x) = x^2 + 6x + 2$

Step 1: Take the individual derivative to find $g'(x) = 2x + 6$

Step 2: Substituting this value into Equation (6) to get

$$f'(x) = \frac{1}{x^2 + 6x + 2} \cdot (2x + 6) = \frac{2x + 6}{x^2 + 6x + 2}$$

11.2 Derivation of a Cobb-Douglas Demand Function Using a Logarithmic Transformation

A demand function expresses the amount of a good a consumer will purchase as a function of commodity prices and consumer income. A Cobb-Douglas demand function is derived by maximizing a Cobb-Douglas utility function subject to the consumer's income.

Example 6 *Given a Cobb-Douglas utility function $U = x^{\alpha} y^{\beta}$ and the budget constraint $P_x x + P_y y = M$, solve for the optimal values of x and y*

Step 1: Begin with a logarithmic transformation of the utility function

$$\ln U = \alpha \ln x + \beta \ln y$$

Step 2: Set up the Lagrangian function

$$\mathscr{L} = \alpha \ln x + \beta \ln y + \lambda (M - P_x x - P_y y)$$

Step 3: Take the first-order conditions

$$\begin{aligned}
\mathscr{L}_x &= \alpha \cdot \frac{1}{x} - \lambda P_x = 0 \Rightarrow \alpha = \lambda P_x x \\
\mathscr{L}_y &= \beta \cdot \frac{1}{y} - \lambda P_y = 0 \Rightarrow \beta = \lambda P_y y \\
\mathscr{L}_\lambda &= M - P_x x - P_y y = 0 \Rightarrow M = P_x x + P_y y
\end{aligned}$$

Step 4: Add $\alpha + \beta$ from $\mathscr{L}_x$ and $\mathscr{L}_y$, recalling that $M = P_x x + P_y y$

$$\alpha + \beta = \lambda(P_x x + P_y y) = \lambda M$$

Step 5: Solve for λ

$$\lambda = \frac{\alpha + \beta}{M}$$

Step 6: Substitute $\lambda = \frac{\alpha+\beta}{M}$ back into $\mathscr{L}_x$ and $\mathscr{L}_y$

$$\alpha = \frac{\alpha + \beta}{M} P_x x \qquad\qquad \beta = \frac{\alpha + \beta}{M} P_y y$$

Step 7: Solve for optimal value of x and y

$$x^* = \left(\frac{\alpha}{\alpha + \beta}\right)\left(\frac{M}{P_x}\right) \qquad\qquad y^* = \left(\frac{\beta}{\alpha + \beta}\right)\left(\frac{M}{P_y}\right)$$

Example 7 *Given the utility function $u = x^{0.3}y^{0.7}$ and the income constraint $M = \$200$, derive the demand functions for goods x and y and evaluate them at $P_x = \$5$ and $P_y = \$8$.*

Step 1: Begin with a logarithmic transformation of the utility function

$$\ln U = 0.3 \ln x + 0.7 \ln y$$

Step 2: Set up the Lagrangian function

$$\mathscr{L} = 0.3 \ln x + 0.7 \ln y + \lambda(200 - 5x - 8y)$$

Step 3: Take the first-order conditions

$$\begin{aligned}
\mathscr{L}_x &= 0.3 \cdot \frac{1}{x} - \lambda 5 = 0 \Rightarrow 0.3 = \lambda 5x \\
\mathscr{L}_y &= 0.7 \cdot \frac{1}{y} - \lambda 8 = 0 \Rightarrow 0.7 = \lambda 8y \\
\mathscr{L}_\lambda &= 200 - 5x - 8y = 0 \Rightarrow 200 = 5x + 8y
\end{aligned}$$

Step 4: Add $0.3+0.7$ from $\mathscr{L}_x$ and $\mathscr{L}_y$, recalling that $200 = 5x+8y$

$$0.3+0.7 = \lambda(5x+8y) = \lambda 200$$

Step 5: Solve for λ

$$\lambda = \frac{0.3+0.7}{200} = \frac{1}{200}$$

Step 6: Substitute $\lambda = \frac{1}{200}$ back into $\mathscr{L}_x$ and $\mathscr{L}_y$

$$0.3 = \frac{1}{200}5x \qquad\qquad 0.7 = \frac{1}{200}8y$$

Step 7: Solve for optimal value of x and y

$$x^* = \frac{0.3(200)}{5} = 12 \qquad\qquad y^* = \frac{0.7(200)}{8} = 17.5$$