Mathematical Derivation of the IS-LM Curve Model

We assume a fixed price level: The price level P is assumed to be fixed in the IS-LM model unless it is specified otherwise. We will also consider a closed economy.

What Are We Trying to Do?

The IS-LM is the most broadly used frame of reference in macroeconomics theory.

In the IS-LM Curve Model, the interactions between the goods (output) market and the financial market gives the equilibrium Y^* and the equilibrium interest rate r^*.

As this new Y^* and r^* satisfy the equilibrium conditions in the goods market as well as the money market.

Note: Different texts use different letters or symbols for the various functions.

We first look at the IS curve (Investment=Savings)

The IS curve show the various combinations of the interest rate (*r*) and income (Y) which satisfy the equilibrium condition in the goods market or make the demand equal to the supply in the goods market.

The Components are:

(1) Consumption Expenditure. The consumption function is specified as

C = a + b (Y-T) where **a** is autonomous consumption and **b** is the marginal propensity to consume.

For now, we assume that $T = T_0$. More realistic assumptions such that $T = T_0 + t_1 Y$ in a more advanced model can be also developed.

(2) Investment Expenditure. The Investment function is specified as

I = I₀ – gr, where $\mathbf{I_0}$ is the autonomous investment and **g** is the reciprocal of the slope of the investment demand curve putting r on the vertical axis and I on the horizontal axis. The larger the value of g, the more elastic is investment with respect to the rate of interest.

The larger the value of **g**, the more responsive is investment with respect to changes in interest rates. In other words, the larger the value of g, the more interest-rate elastic is investment. A numerical example would be $I = 100 – 5r$. A one percentage point increase in the rate of interest investment will bring about a 5 percentage point decrease in investment.

(3) Government Expenditure = G_0, where **G_0** is independent of the level of Y and r.

IS Curve: Goods Market Equilibrium

Algebraic Derivation

Suppose that we are dealing with the aggregate expenditures with only lump-sum taxes and no exports or imports. In equilibrium Y = Aggregate Expenditure (AE)

The AE will be

$$\mathbf{AE = a + b\,(Y - T_0) + I_0 - gr + G_0} \quad \text{or} \quad \mathbf{AE = a + b\,(Y - T_0) + I_0 + G_0 - gr}$$

At equilibrium, Y= AE

Solve for Y* and *r**: We can rewrite this equation as a functional relationship between Y and the interest rate *r*.

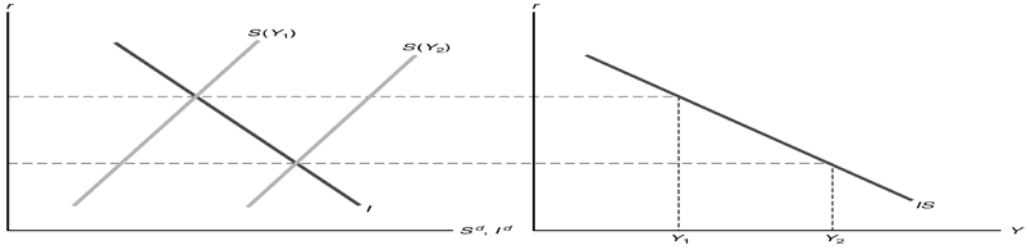
$$Solving\ for\ Y:$$

$$\begin{aligned} Y &= a + b(Y - T_0) + I_0 + G_0 - gr \\ Y &= a + bY - bT_0 + I_0 + G_0 - gr \\ Y - bY &= a - bT_0 + I_0 + G_0 - gr \\ Y(1-b) &= a - bT_0 + I_0 + G_0 - gr \\ Y &= \frac{1}{1-b}(a - bT_0 + I_0 + G_0 - gr) \end{aligned}$$

$$Solving\ for\ r:$$

$$r = \frac{1}{g}(a - bT_0 + I_0 + G_0) - \frac{1-b}{g}Y$$

- In economics, it is customary to put the price or interest rate on the vertical axis, and the quantity on the horizontal axis when drawing a graph or a diagram. So the second one is in line with the illustrative tradition. When we draw **the *IS curve* we put *r* on the vertical axis and Y on the horizontal axis.**



We can solve for the slope of the IS curve

$$From\ before:$$
$$r = \frac{1}{g}(a - bT_0 + I_0 + G_0) - \frac{1-b}{g}Y$$
$$Solving\ for\ \frac{dr}{dY}:$$
$$\frac{dr}{dY} = -\frac{1-b}{g}Y$$

Solving for the slope of the IS curve using calculus.

$$Y = a + b(Y - T_0) + I_0 + G_0 - gr$$
$$Taking\ total\ differential:$$

$$dY = bdY - gdr$$
$$(1-b)dY = gdr$$

$$Solving\ for\ the\ slope\ of\ IS:$$
$$\frac{dr}{dY} = -\frac{1-b}{g}$$

- Note that the slope has a negative sign and thus the IS curve is downward sloping. This means that in equilibrium in the goods market, the interest rate and income move in the opposite direction; if the interest rate increases for some reason, in order to stay at the same equilibrium in the goods market, national income should decrease

- You can show that the larger the value of g (more elastic is investment with respect to the rate of interest), the flatter the IS curve.

- A change in a, I_0, or G_0 changes Y by the amount of the multiplier, which is 1/1-b.

- A change in T_0, changes Y by -b/1- b.

$$\frac{\partial Y}{\partial a} = \frac{\partial Y}{\partial I_0} = \frac{\partial Y}{\partial G_0} = \frac{1}{1-b} \qquad \frac{\partial Y}{\partial T_0} = \frac{-b}{1-b}$$

We now look at the LM Curve (Liquidity Preference = Money Supply)

The LM cure shows the various combinations of the interest rate (*r*) and income (Y) which satisfy the equilibrium condition in the money market.

(1) 'Nominal' versus 'Real' Money Supply/Demand

We should make distinction between Nominal Supply or Demand and Real Supply or Demand. The first one is in monetary terms, and the second in quantity terms.

Real quantity of money = Nominal quantity of money/Price level.

$m = M/P$

(2) Money Supply

The *nominal quantity* of the money supply is determined by the monetary authority, which usually is the central bank.

$MS = M_{0 =}$ constant amount at a point of time.

We regard the nominal money supply as an exogenous variable, and regard it as arbitrarily determined by the monetary authority.

Mathematically, this means that the nominal money supply curve is vertical, being independent of interest rates

At one point of time it is fixed. However, of course, over time it can be changed by the monetary authority. In fact, the monetary authority sets the nominal money supply in each period.

(3) Real Money Demand

The real money demand is given a functional form such as

$m^d = M^d/P = L(Y, r).$

The money demand is called' *liquidity preference'*, and the money demand function '*liquidity preference function.*'

Since . $m^d = L(Y, r)$. *and we look at* $L_1 = L_1(Y)$ *and* $L_2 = L_2(r)$

We can give the liquidity preference function the following specific functional form;

$\boldsymbol{m^d = kY - hr,}$ where *k* is the elasticity of real money demand with respect to the national income and *h* is the elasticity of real money demand with respect to interest rates.

The liquidity preference curve is negatively sloped when drawn with the interest rate on the vertical axis and the amount of real money on the horizontal axis. The variable Y is the shift parameters of the real money demand curve.

LM Curve: Money Market Equilibrium

We are assuming the price level is fixed so we can set it at 1. Thus, the $m^s = M^s/1 = M^s = m^d = M^d/1 = M^d$ or real supply = real demand for money.

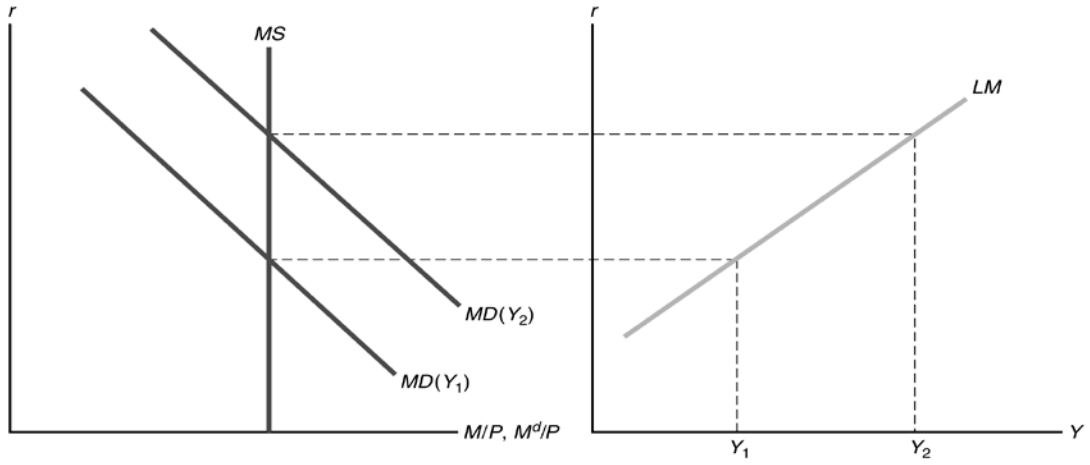
Algebraic Derivation

LM Curve: Money market equilibrium condition $M^s = M^d$ yields the following equations. And then you can solve for Y* or r*.

$$M^s = kY - hr$$
$$kY = M^s + hr$$
$$Y = \frac{1}{k}M^s + \frac{h}{k}r$$
$$Solving\ \ for\ r:$$

$$hr = -M^s + ky$$
$$r = -\frac{1}{h}M^s + \frac{k}{h}Y$$

- When we draw **the LM *curve* we put *r* on the vertical axis and Y on the horizontal axis.**



We can solve for the slope of the LM curve.

$$From\ before:$$

$$r = -\frac{1}{h}M^s + \frac{k}{h}Y$$

$$Solving\ for\ \frac{dr}{dY}:$$

$$\frac{dr}{dY} = \frac{k}{h}$$

Solving for the slope of the IS curve using calculus.

$$M^s = kY - hr$$

$$Taking\ total\ differential:$$

$$dM^s = kdY - hdr$$

$$M^s\ is\ constant, so\ dM^s = 0$$

$$Solving\ \ for\ the\ slope\ of\ LM:$$

$$\frac{dr}{dY} = \frac{k}{h}$$

- Note that the slope has a positive sign and thus the LM curve is upward sloping. This means that in equilibrium in the money market, the interest rate and income move in the same direction; if the interest rate increases for some reason, in order to stay at the same equilibrium in the money market, national income should increase

- You can see that the larger the value of h (more elastic is the demand for money with respect to the rate of interest), the flatter the LM curve. If the demand for money is perfectly elastic, the LM curve is flat. We refer to this as the liquidity trap.

- If the demand for money is relatively inelastic, the LM curve is steep.

- The smaller the value of k, the flatter the LM Curve.

- The shape of the LM curve has implications for monetary and fiscal policy.