INTERMEDIATE MICROECONOMICS LECTURE 8 - THEORY OF PRODUCTION

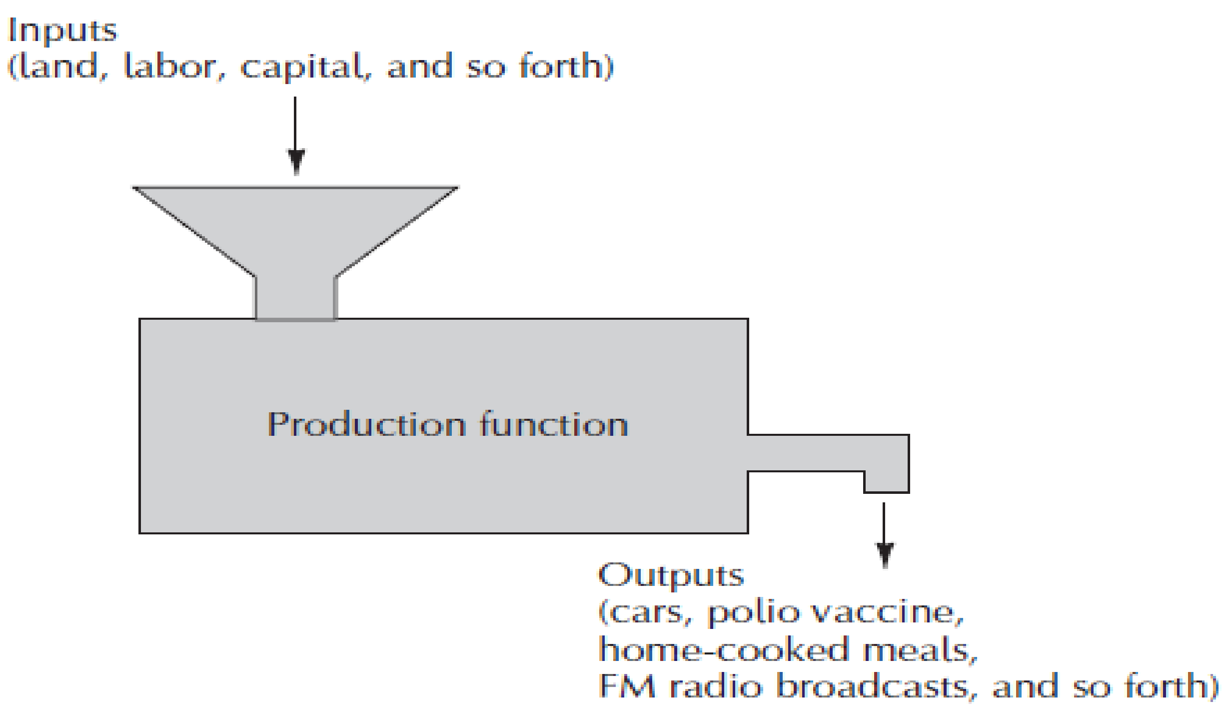
Production is the entire process of making goods and services. In theoretical terms production involves the transformation of inputs into outputs, and in practice involves input sourcing, capital equipment acquisition, industrial engineering, adaptation of new technology, personnel training and management, and a great deal of managerial coordination. Production is a firm's response not only to the consumer market signals of 'what' type of good or service is demanded, but also to the input market signals of 'how' to produce, given the costs associated with various labor and capital/technological inputs.

Production Functions

A *production function* is a formal mathematical relation that describes the efficient process of transforming inputs into outputs. The word efficient is in the definition because embedded in the concept of the production function is the notion that firms will want to squeeze the maximum output from a given set of inputs.

A simple production function describes the process of transforming a set of inputs K, L, etc. into a quantity Q of output (goods or services): Q = Q(K, L etc.)

Conceptually, a specific production function, Q(K, L), defines a technology. This is extremely similar to a recipe in so many ounces of flour, liquid, and sweetener, properly combined then baked, creates a cake or so many ounces of steel, fasteners, rubber, and glass, properly combined, creates an automobile. Different recipes to make cakes and automobiles represent different technologies.



Production Decisions in the Short – Run

The short run is defined as the time frame in which there are fixed factors of production

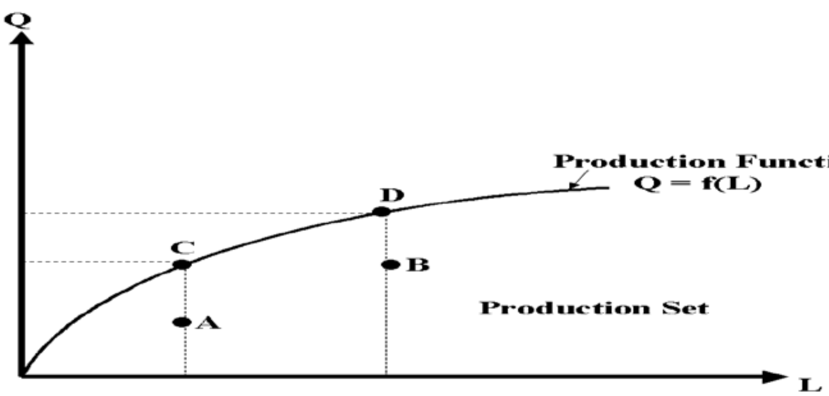
Variable and fixed factors of production – variable factors are the inputs a manager can adjust to alter production. Fixed factors are the inputs the manager cannot adjust for a period of time

Usually capital (K) is fixed (denoted by $\overline{K}$) in the short run and labor (L) is variable, so we can rewrite the production function as,

$Q = f(L) = F(\overline{K}, L)$

Total, Marginal, and Average Product

Total product (TP) is the entire output of the production process, and often denoted as the Q, or quantity of output.



Marginal product (MP) of a particular factor of production, for example labor, is defined to be the change in output (Q) resulting from a one-unit change in the input (ex: one more hour worked, or one more worker employed). If again we use the symbol Δ to mean 'change in', then we have:

$MP_X = \frac{\Delta Q}{\Delta X}$ = where X is a particular input, such as labor, and Q is output of the final good or service produced. Marginal product is the slope of total product.

If we have a nice continuous total product curve, then marginal product at a particular point on the total product curve is simply the slope of the tangent line at that point on the total product curve.

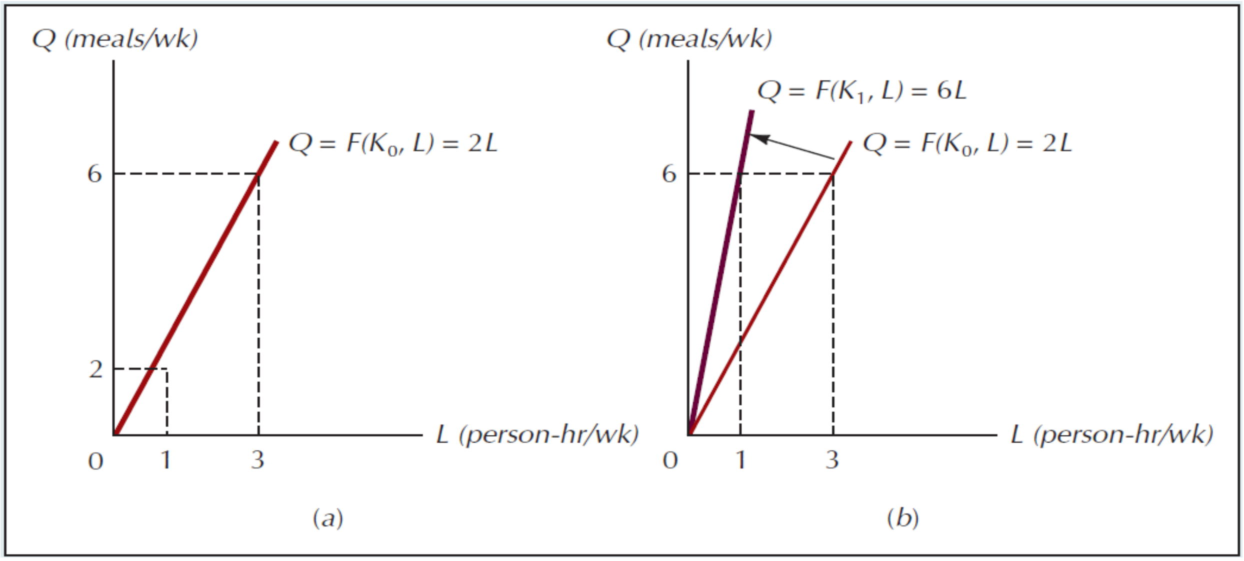
Using calculus $MP_X = \frac{\partial Q}{\partial X}$

Average product (AP) is the average amount of output produced with each unit of input:

$AP_X = \frac{Q}{X}$

When we have a nice, continuous total product curve, average product at a particular point on the total product curve is the slope of a ray that comes out of the origin of the diagram and passes through the point in question on the total product curve.

A Specific Short-Run Production Function



In the diagram above what are the values of MP_L and AP_L ?

Diminishing Marginal Returns

Very generally, in the short run (when, for example, the production facility is fixed in size) there are two intuitive processes at work:

Gains in marginal productivity as we add more of a variable factor of production (ex: workers) due to specialization

Declines in marginal productivity as we add more of a variable factor of production due to congestion in the fixed factor (ex: a kitchen).

The concept of ***diminishing marginal returns*** comes from empirical observation of actual production processes. The idea is intuitive: as more and more of a variable factor (X) is combined with a fixed factor (Y), the marginal productivity of the variable factor, $\Delta Q/\Delta X$, eventually will decline. The central motivation for diminishing marginal returns to a variable factor is that the fixed factor will eventually become *congested* with the variable factor. As we keep adding more and more kitchen employees to a kitchen of fixed capacity, the space will become congested with kitchen workers and eventually there will not be enough capital (ex: pots, pans, space on the cooker, utensils, ovens) for each additional worker to be as productive as the previous worker added. Moreover, it is possible that the congestion can become so acute that hiring an additional worker will actually *lower total product*, implying a negative marginal product.

In the general case we do not think that diminishing marginal returns sets in immediately because there frequently are economies of specialization, especially for labor as a variable factor. For example, as we initially hire a second kitchen worker, the two kitchen workers can now specialize -- one preps food ingredients, the other combines ingredients (cooks them) to make final meals.

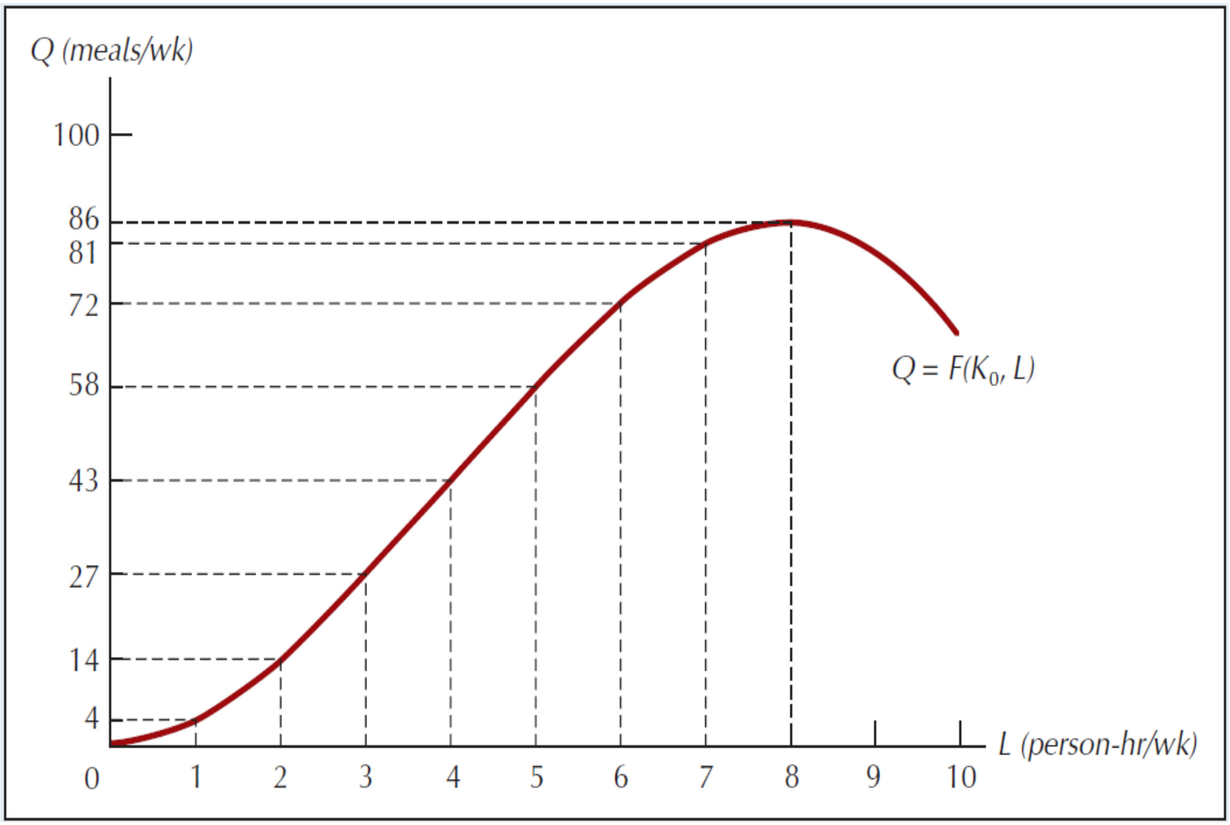
Thus we might make a generalization for the purposes of illustration of the concept of marginal productivity that there are three stages of production:

Stage 1: Increasing marginal returns

Stage 2: Diminishing marginal returns

Stage 3: Negative marginal returns

Let's illustrate a very simple production function. Let the fixed input be the size of the kitchen. Let the variable input be the number of workers that we apply to the kitchen to make meals.



Knowing total output and the corresponding input levels, we can solve for the Average Product and Marginal Product of Labor. And note again that we are measuring everything in physical units and that only the variable input is allowed to vary.
Note that AP_L and MP_L both pass through a maximum and then decline. This shows the **LAW OF DIMINISHING MARGINAL RETURNS** (law of diminishing marginal productivity or law of diminishing returns) discussed above.

Looking at Some Numbers

Average Product, Total Product, and Marginal Product (lb/day) for Two Fishing Areas

Number of boats	East end AP	East end TP	East end MP	West end AP	West end TP	West end MP
0	0	0		0	0	
			100			130
1	100	100		130	130	
			100			110
2	100	200		120	240	
			100			90
3	100	300		110	330	
			100			70
4	100	400		100	400	

The average catch per boat is constant at 100 pounds per boat for boats sent to the east end of the lake. The average catch per boat is a declining function of the number of boats sent to the west end.

Now for some real fun, let's see how we can derive the shapes of the AP and MP curves via some graphical constructions (see graph on next page):

The Relationships between Total Product and the Average and Marginal Product of the Variable Input (general case)

1. When *TP* is at a maximum, $MP_X = 0$.
2. When the AP_X is increasing, then $MP_X > AP_L$.
3. When AP_X is at a maximum, $MP_X = AP_X$.
4. When the AP_X is declining, then $MP_X < AP_X$.

CAN YOU PROVE THESE STATEMENTS?

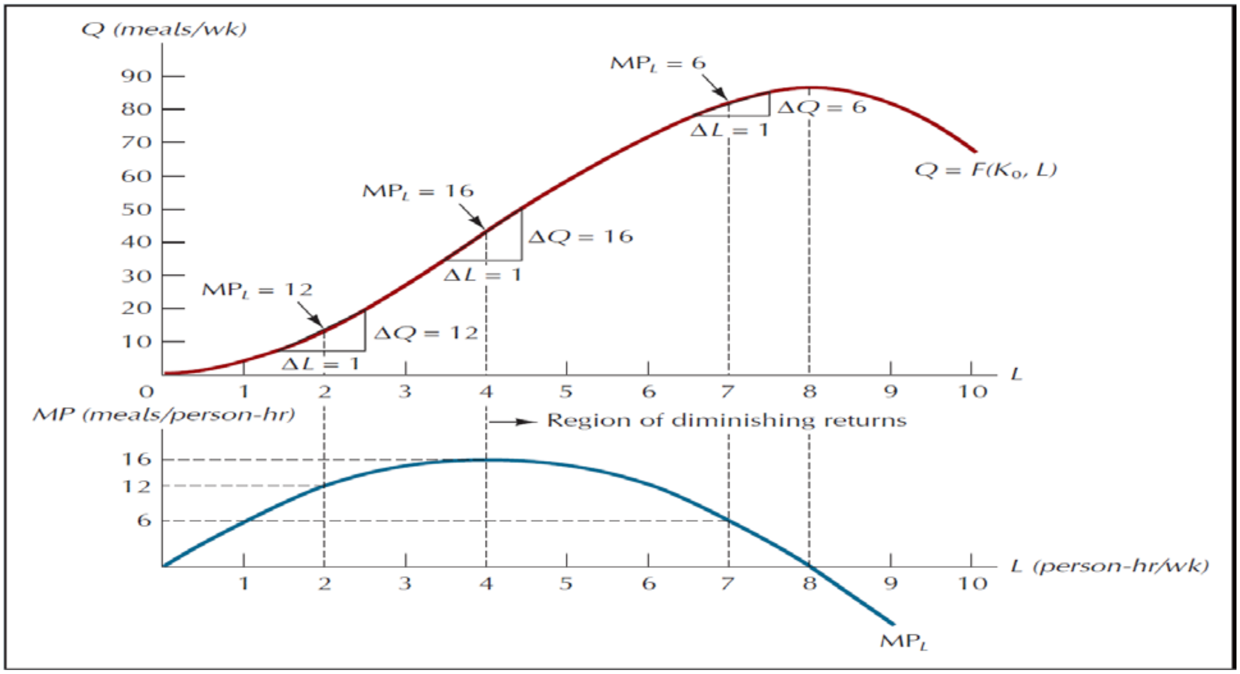
$$AP_L = \frac{TP}{L}$$

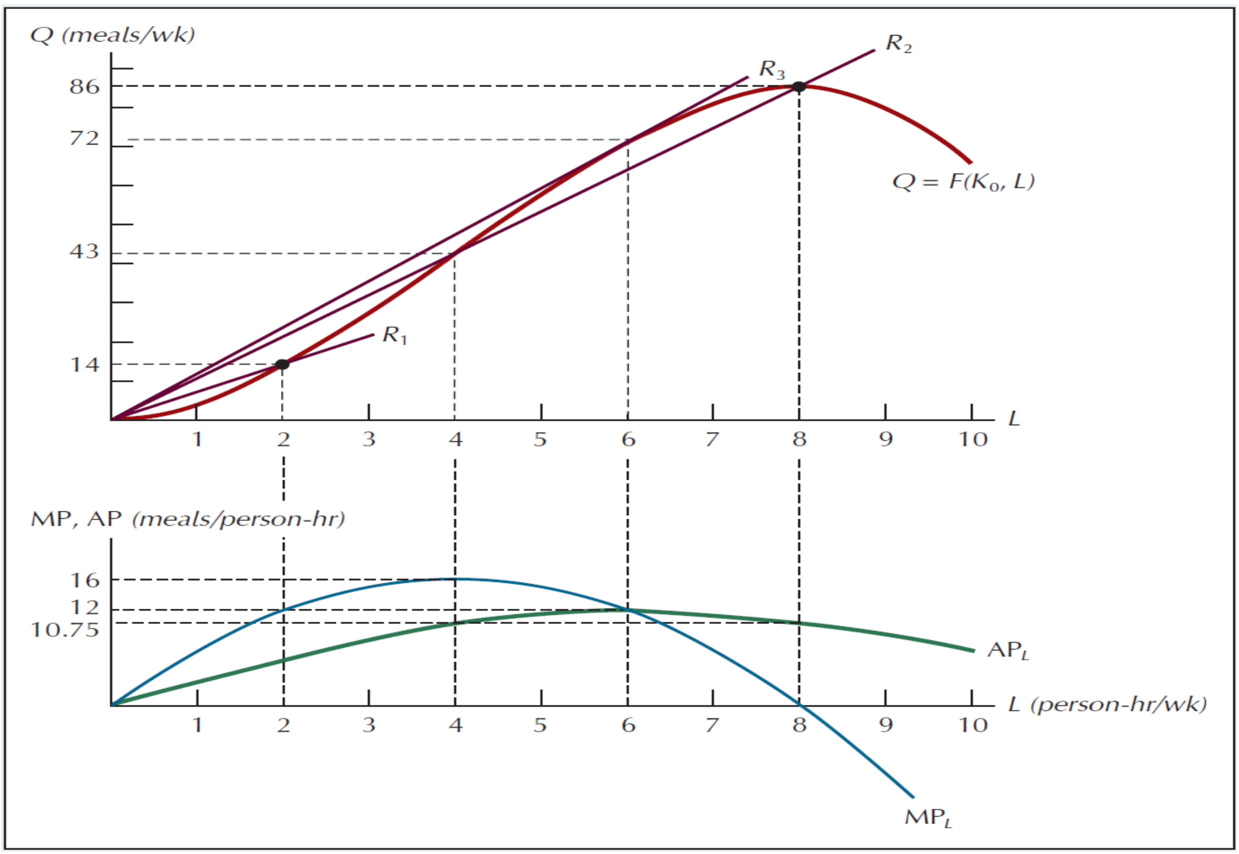
$$\frac{dAP_L}{dx} = \frac{d\,\dfrac{TP}{L}}{dL} = \frac{L\dfrac{dTP}{dL} - TP\dfrac{dL}{dL}}{L^2} = \frac{(LMP_L - TP)}{L^2}$$

$$Multiply\ by\ \frac{\frac{1}{L}}{\frac{1}{L}} = \frac{(MP_L - AP_L)}{L}$$

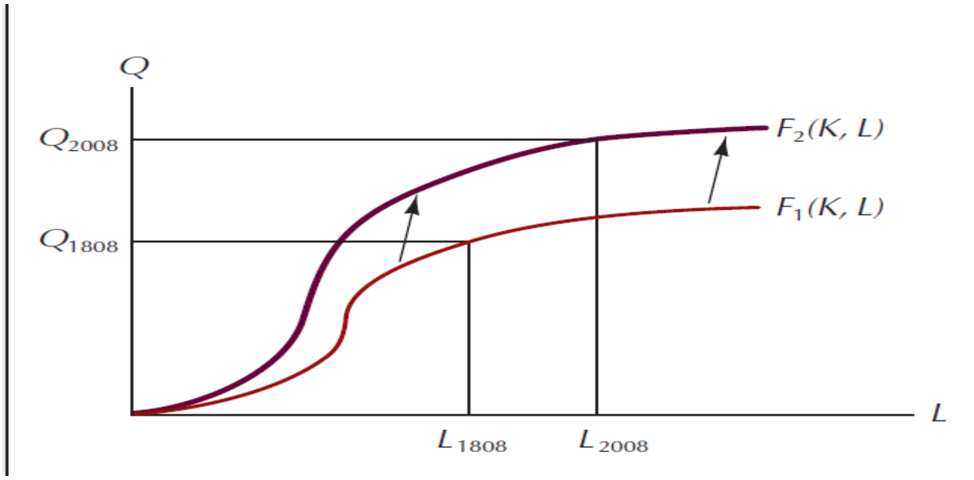
$$\frac{dAP_L}{dL} = \frac{(MP_L - AP_L)}{L}$$

$$You\ can\ show\ that\ if\ MP_L > AP, \frac{dAP_L}{dL} > 0\ \ or\ upward\ sloping$$


Q (meals/wk)
90
80
70
60
50
40
30
20
10
MP_L = 6
ΔQ = 6
ΔL = 1
Q = F(K_0, L)
MP_L = 16
ΔQ = 16
ΔL = 1
MP_L = 12
ΔQ = 12
ΔL = 1
0 1 2 3 4 5 6 7 8 9 10
L
MP (meals/person-hr)
Region of diminishing returns
16
12
6
1 2 3 4 5 6 7 8 9 10
L (person-hr/wk)
MP_L


Q (meals/wk)
86
72
43
14
R_2
R_3
R_1
Q = F(K_0, L)
1 2 3 4 5 6 7 8 9 10
L
MP, AP (meals/person-hr)
16
12
10.75
AP_L
1 2 3 4 5 6 7 8 9 10
L (person-hr/wk)
MP_L

The Effect of Technological Progress in Food Production



Production Planning When Factors are Variable: Isoquants and the Marginal Rate of Technical Substitution

An *isoquant* is a line on a two-input diagram (y input on y axis, x input on x axis) that shows equal levels of output that can be generated by different combinations of the two inputs. Implicit in the isoquant is the notion that there is *efficient production*, meaning that the points along an isoquant correspond to the *maximum* output possible from the particular input combination. Another condition is commonly referred to as *technical efficiency*, because it relates to least-cost production methods or technology for producing output.

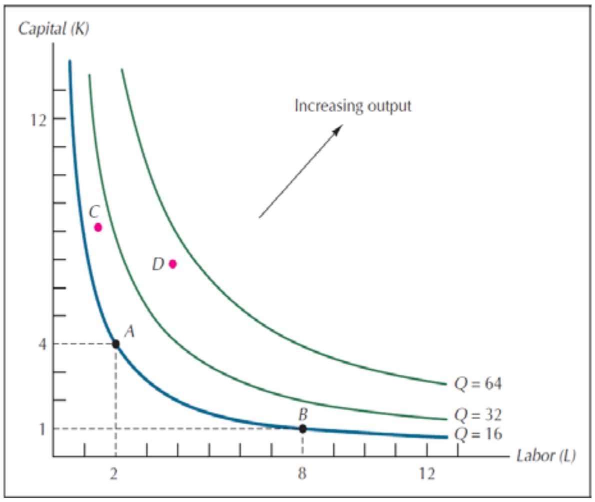
Isoquants are to production what indifference curves are to utility.

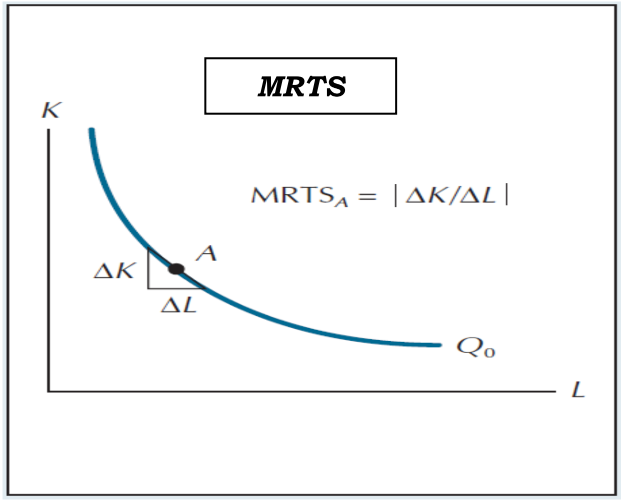
Movement along an isoquant illustrates the concept of *input factor substitution*, meaning the extent to which one input (say, for example, labor) can substitute for another input (say, for example, a machine in producing a given output

Note that higher isoquants imply higher levels of output. Isoquants are similar in many ways to indifference curves, and serve a role in optimal input combination selection as indifference curves do in optimal consumption bundle selection.

Movement along the isoquant involves:

- A change in the input combination
- The same level of total output
- Thus movement along the isoquant involves input substitution -- as more of one input is used, less of the other *must* occur in order to keep total output constant.





The ***Marginal Rate of Technical Substitution (MRTS),*** shown above, is the slope of the isoquant, and tells us how much of the input on the 'y' axis we must give up in order to use more of the input along the 'x' axis. Thus in the isoquant diagram above, the MRTS relates the number of units of capital that must be reduced as one utilizes more laborers in order to keep the output constant. Using the symbol Δ to denote 'change in', we have:

MRTS = $\Delta K/\Delta L$ = slope of the isoquant

Where K and L are inputs in the production process and since movement along the isoquant holds Q constant, we also can show that:

$MP_L \Delta L + MP_K \Delta K \equiv 0$ Rearranging $MP_L \Delta L = - MP_K \Delta K$.

Remember, marginal rate of technical substitution is the slope of the isoquant, which is $\Delta K/\Delta L$ (rise over the run). So as you can see

MRTS = $-\Delta K/\Delta L = MP_L/MP_K$. If we use calculus, the production function is given as:

Q = Q(K, L)   where "Q" is a constant level of output along an isoquant.

Take the total derivative of this isoquant and set it equal to zero (Why?):

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL = 0$$

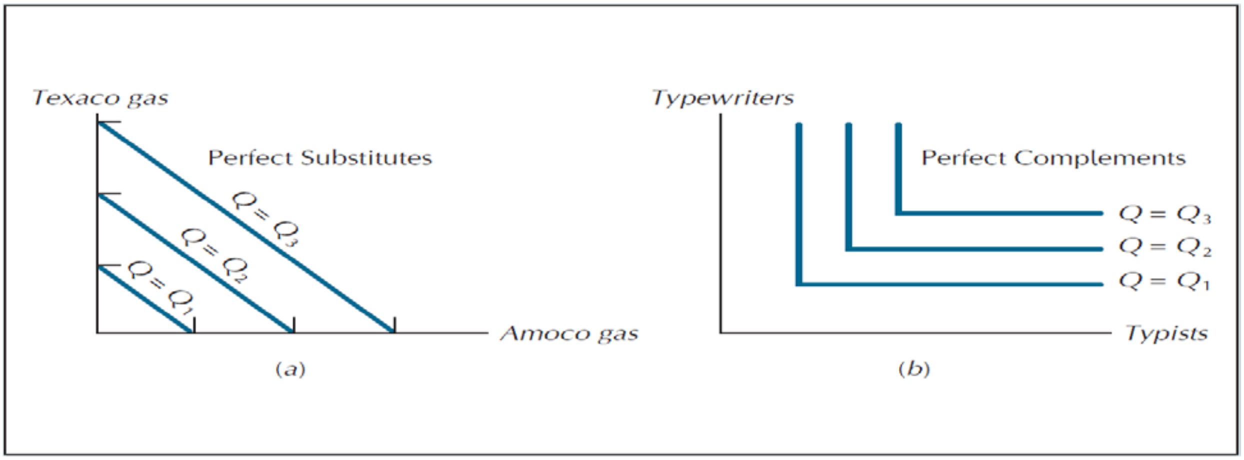
Solving for the slope of the isoquant:

$$-\frac{dK}{dL} = MRTS_{L,K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} \quad \text{or} \quad MRTS_{L.K} = \frac{MP_L}{MP_K}$$

So the marginal rate of technical substitution is simply the ratio of the marginal products.

Recall the law of diminishing marginal returns. As we use more and more of the 'x' input, and less and less of the 'y' input, the marginal product of the 'y' input rises while the marginal product of the 'x' input falls, meaning that the isoquant for imperfect substitutes becomes *flatter* as one moves down along the isoquant using more 'x' and less 'y'. When inputs are *perfect complements*, such as wheels and chassis for automobiles, the isoquant has a 'square L' shape (why?).

When inputs are perfect substitutes, such as Farmer Dan's rice versus Farmer Mary's rice the isoquant is a straight line (why?).

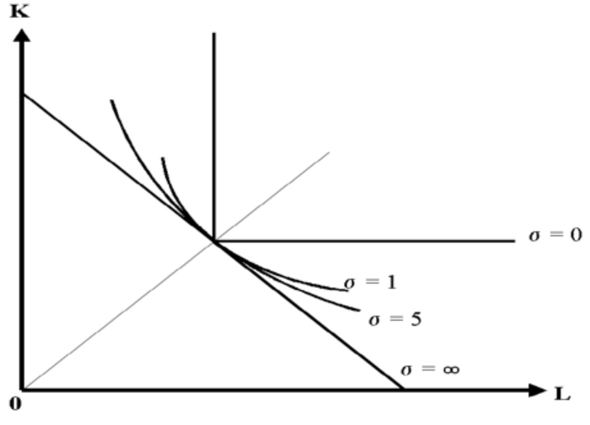


Definition: The **Elasticity of Substitution**,

$$\sigma = [\frac{\Delta(K/L)}{\Delta(MRTS_{L,K})} x \frac{MRTS_{L,K}}{(K/L)}] \quad \text{or} \quad \sigma = [\frac{\Delta(K/L)}{\Delta(MP_L/MP_K)} x \frac{MRTS_{L,K}}{(K/L)}]$$

measures how the change in the capital-labor ratio, K/L, changes relative to the change in the $MRTS_{L,K}$.

"The shape of the isoquant indicates the degree of substitutability of the inputs…"



1) The Linear Production Function – a production function that assumes a perfect linear relationship between all inputs and total output.

Q = F (K, L) = *a*K + *b*L

The linear nature of this production function implies that the inputs can be used as perfect substitutes; therefore which one was used would depend on the productivity of each input and input cost

Using calculus to find the MP of capital and labor

$$MP_K = \frac{\partial Q}{\partial K} = a \qquad MP_L = \frac{\partial Q}{\partial L} = b$$

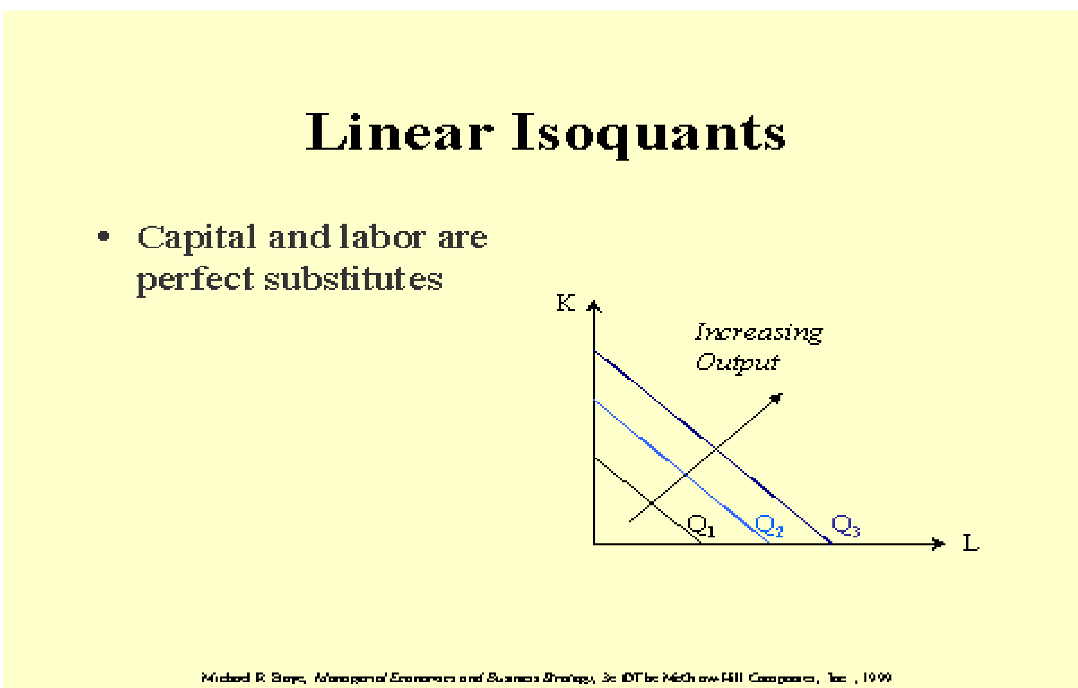
EXAMPLE: Q = 4K + 3L

What is the marginal product of each input? $MP_L = 3$; $MP_K = 4$

If output is 1000 units, how much labor can we use if we don't use any capital in the production process? L = 333 (what about no labor then K = 250)

Which input is more productive? K because $MP_K = 4$

If Q = F (6,4) how much can be produced? 36 units



2) Leontief Production Function – a production function that assumes that inputs are used in fixed proportions, or a certain amount of K must be used with L, NO SUBSTITUTABILITY of the inputs.

$Q = F(K, L) = \min\{bK, cL\}$

- the min implies that we will produce the lower of the two inputs

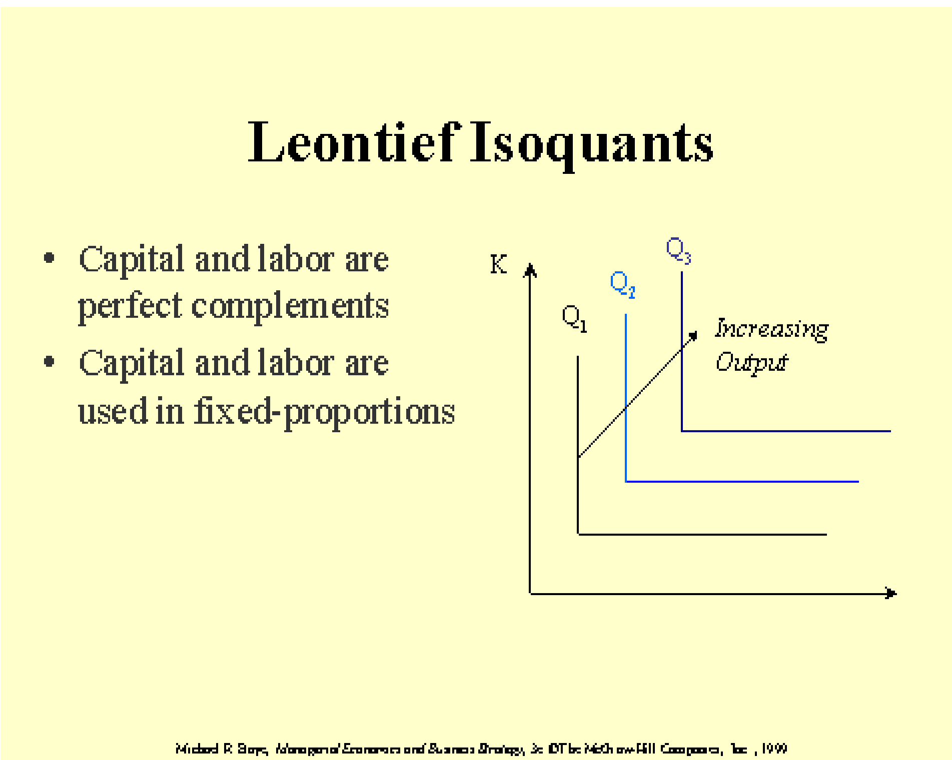
EXAMPLE:

$Q = \min\{3K, 4L\}$

How much output will be produced if we use 3 units of labor and 6 units of capital?
- $4L = 4 \times 3 = 12$ units
- $3K = 3 \times 6 = 18$ units

- therefore 3 units of labor and 6 units of capital will produce the smaller of 12 & 18 so only 12 units

Look at the example of a firm that is responsible for street cleaning. Let output be the area cleaned and Q = 1. The production function is Q = min{broom, workers}, since each broom requires one person to operate it and we only have one worker how many brooms are needed? ONLY 1



3) Cobb Douglas Production Function – a production function that assumes some degree of substitutability between inputs.

In general form it may be written as $Q = K^{\alpha} L^{\beta}$ or $Q = K^{\alpha} L^{1-\alpha}$ (special case)

The inputs are not perfect substitutes and don't need to be used in fixed proportions.

The exponents tell the productivity as share of output, for example Q = $K^{1/2} L^{1/2}$ means that L & K each do half of output, Q = $K^{2/3} L^{1/3}$ means that K is twice as productive as L.

EXAMPLE: Q = $K^{1/2} L^{1/2}$

What is Q if K = 36 and L = 36 Q = $(36)^{1/2} (36)^{1/2}$ = (6)(6) = 36

Using calculus to find the MP's (general form of Cobb-Douglas function).

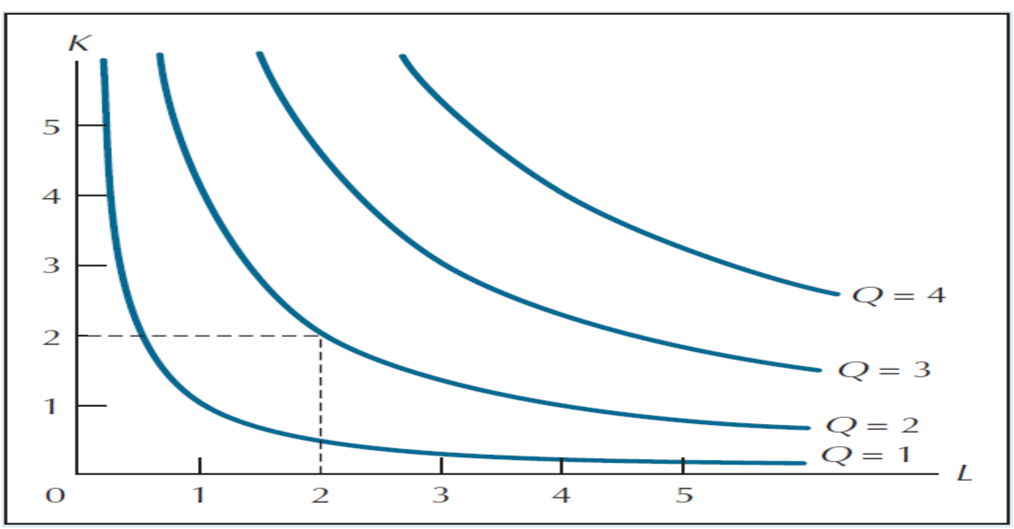
$$Q = K^{\alpha} L^{\beta}$$

$$MP_K = \frac{\partial Q}{\partial K} = \alpha K^{\alpha-1} L^{\beta} \qquad MP_L = \frac{\partial Q}{\partial L} = \beta K^{\alpha} L^{\beta-1}$$

From above we can calculate the MRTS.

MRTS = $\dfrac{MP_L}{MP_K} = \dfrac{\beta K^{\alpha} L^{\beta-1}}{\alpha K^{\alpha-1} L^{\beta}} = \dfrac{\beta K}{\alpha L}$. This is constant.

Isoquant Map for the Cobb-Douglas Production Function Q = $K^{1/2} L^{1/2}$



The Cobb-Douglas Production Function (two-input case)

$$Q = mK^{\alpha}L^{\beta}, \quad (8A.3)$$

where α and β are numbers between zero and 1, and m can be any positive number.

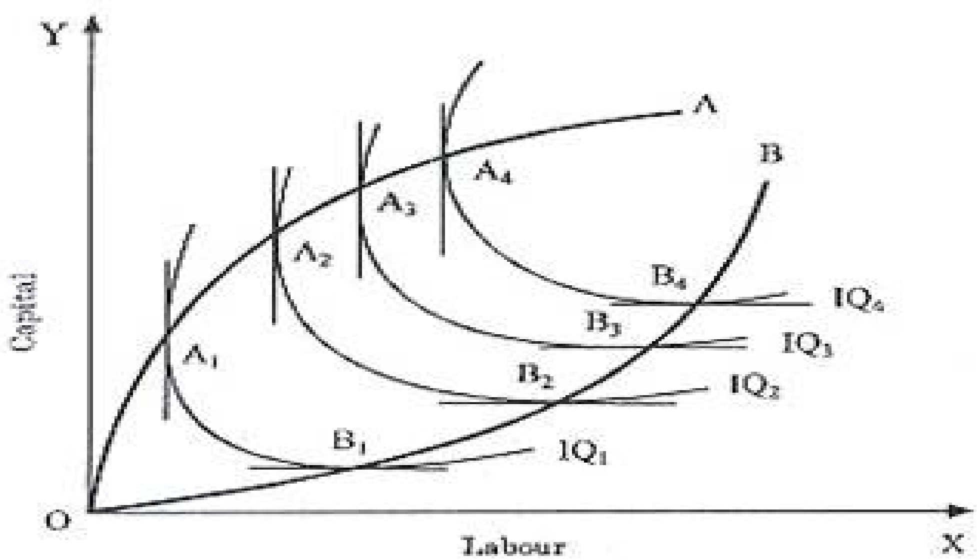
To generate an equation for the Q_0 isoquant, we fix Q at Q_0 and then solve for K in terms of L. In the Cobb-Douglas case, this yields

$$K = \left(\frac{m}{Q_0}\right)^{-1/\alpha}(L)^{-\beta/\alpha}. \quad (8A.4)$$

For the particular Cobb-Douglas function $Q = K^{1/2}L^{1/2}$, the Q_0 isoquant will be

$$K = \frac{Q_0^2}{L}. \quad (8A.5)$$

Ridge Lines – Graphic bounds for positive marginal products. It is irrational for a firm to combine resources in such a way that the marginal product of any input is negative implying output could be increased by using less of the resource. Therefore, we are only concerned with the negatively sloped portion of the production isoquant. The positive sloped portions are irrational.



(a)

Returns to Scale

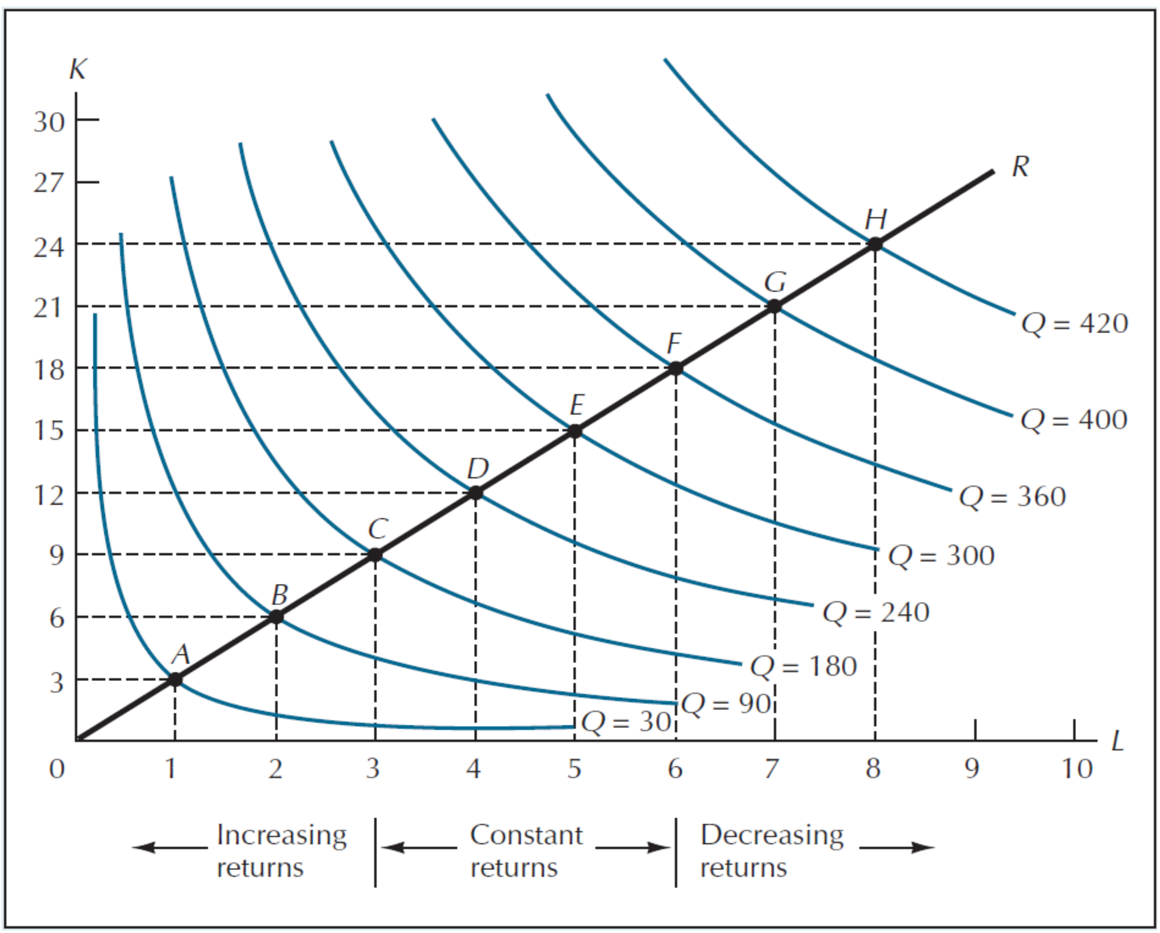
Suppose the firm is considering expansion of its facilities, and the question is how much additional output will be generated by, say, a fully proportionate doubling of all inputs. This question addresses the issue of *returns to scale*. Production processes may feature *increasing, constant, or decreasing returns to scale* over a particular range of expansion, and moreover may experience two or more of these over the entire range of production.

Increasing Returns to Scale: Output increases more than proportionately with an increase in all inputs. For example, a doubling of all inputs will result in a more than doubling of output.

Constant Returns to Scale*:* Output increases proportionately with an increase in all inputs. For example, a doubling of all inputs will result in a doubling of output.

Decreasing Returns to Scale*:* Output increases less than proportionately with an increase in all inputs. For example, a doubling of all inputs will result in a less than doubling of output.

Returns to scale helps the firm determine its optimal size, and thus is naturally a long-run question in microeconomics theory.



Cobb-Douglass Production Function and the Concept of Returns to Scale

From before we said that the general form of Cobb-Douglas function is:

$Q = K^{\alpha} L^{\beta}$ or $Q = K^{\alpha} L^{1-\alpha}$. We can also look at function such as $Q = AK^{\alpha} L^{\beta}$, where

A is a parameter representing technology.

Looking at α and β, we can interpret these as the elasticity of output with respect to input.

$\alpha = \frac{\partial Q}{\partial K}\frac{K}{Q}$ $\beta = \frac{\partial Q}{\partial L}\frac{L}{Q}$ Notice that the value of the elasticity is $\frac{MP}{AP}$

Example of Elasticity:

Let $Q = K^{\alpha} L^{\beta}$ $\frac{\partial Q}{\partial K} = \alpha K^{\alpha-1} L^{\beta}$ $\frac{\partial Q}{\partial K}\frac{K}{Q} = \alpha K^{\alpha-1} L^{\beta}\frac{K}{Q} = \alpha$

$\frac{\partial Q}{\partial L} = \beta K^{\alpha} L^{\beta-1}$ $\frac{\partial Q}{\partial L}\frac{L}{Q} = \beta K^{\alpha} L^{\beta-1}\frac{L}{Q} = \beta$

Returns to scale are defined for homogeneous production functions. We say that a production function is homogenous to degree *t,* if

f(*m*K, *m*L) = m^t f(K,L) where *t* and *m* are constant and greater than 0.

If *t* > 1, we have increasing returns to scale.
If *t* = 1, we have constant returns to scale.
If *t* < 1, we have decreasing returns to scale.

Example:

$Q = K^{\alpha} L^{\beta}$ Let *m* = 2

Define $Q' = Q'(2K, 2L)$ $Q' = Q'(mK, mL)$

$$Q' = (2K)^{\alpha}(2L)^{\beta} = 2^{\alpha} K^{\alpha} 2^{\beta} L^{\beta} = 2^{\alpha+\beta} K^{\alpha} L^{\beta} = 2^{\alpha+\beta} Q$$

If α + β = 1 and we know that $Q' = m^t Q = 2^1 Q$ so this function is homogenous to degree 1, since t=1.